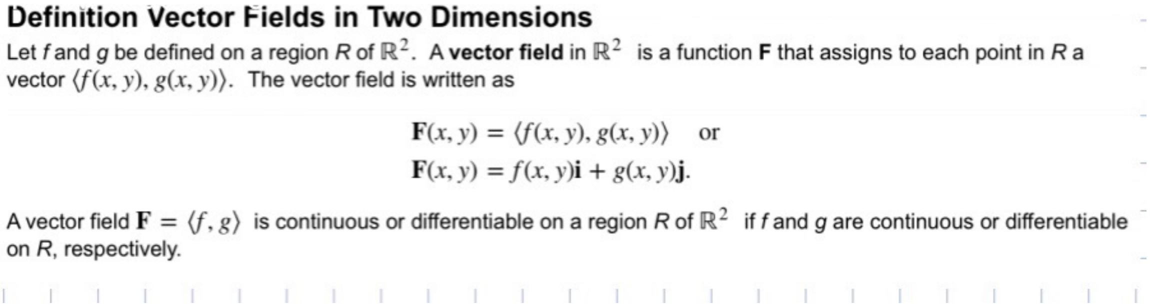
17.1 Vector Fielel.



Definition Vector Fields in Two Dimensions

Let *f* and *g* be defined on a region *R* of $\mathbb{R}^2$. A **vector field** in $\mathbb{R}^2$ is a function **F** that assigns to each point in *R* a vector $\langle f(x, y), g(x, y)\rangle$. The vector field is written as

$$\mathbf{F}(x, y) = \langle f(x, y), g(x, y)\rangle \quad \text{or}$$
$$\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}.$$

A vector field $\mathbf{F} = \langle f, g\rangle$ is continuous or differentiable on a region *R* of $\mathbb{R}^2$ if *f* and *g* are continuous or differentiable on *R*, respectively.

$$F(a.b) = \langle \underset{\uparrow}{\underline{f(a.b)}} , \underset{\uparrow}{\underline{g(a.b)}} \rangle$$

x- component of the vector of the point(a.b)

y-component of the vector of the point (a.b).

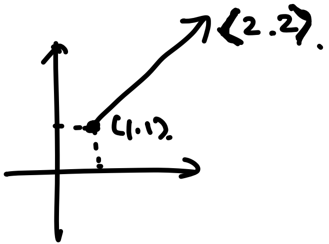
at the point (a.b), we draw vector $\langle f(a.b). g(a.b)\rangle$.

e.g.

$a=1 \quad b=1$

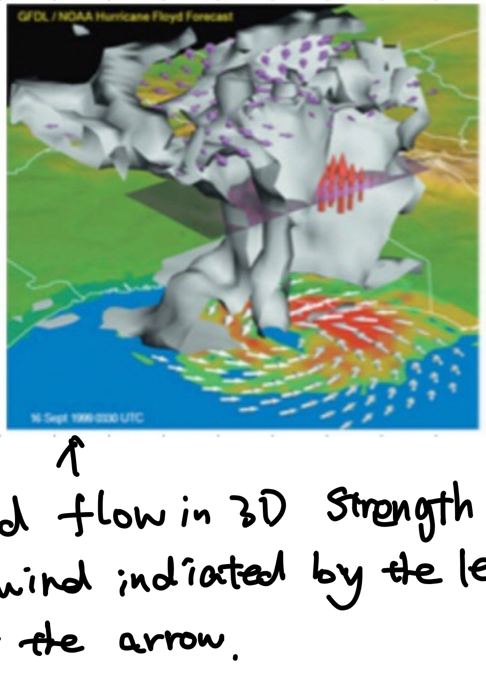
$f(a.b)=2a \quad g(a.b)=2b.$

$f(1.1)=2 \quad g(1.1)=2$

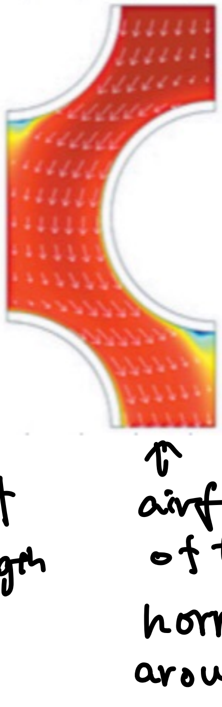


$F(1.1) = \langle 2.2\rangle.$

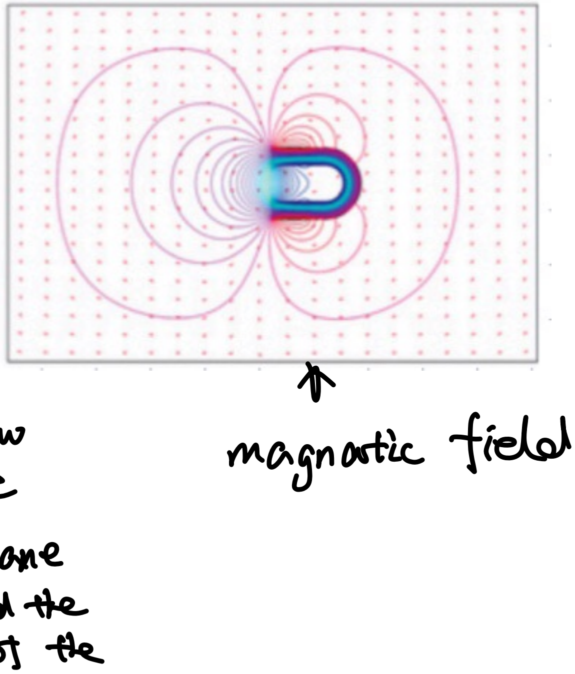
* A lot of what we will learn in chapter 17 is used in different physics & engineering classes. (ch17 is generalization of those class)



↑ wind flow in 3D Strength of the wind indicated by the length of the arrow.



↑ airflow of the horricane around the eye of the hurricane



↑ magnatic field.

ex). $F = \langle 2x, 2y \rangle$.   * Plug in a coordnate in $\mathbb{R}^2$ and get a $\underline{\underline{\text{vector}}}$ back.

$F(0,1) = \langle 2\cdot 0, 2\cdot 1 \rangle = \langle 0,2 \rangle$.

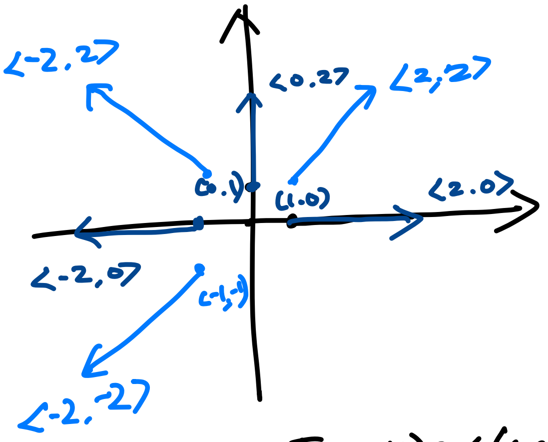
$F(1,0) = \langle 2\cdot 1, 2\cdot 0 \rangle = \langle 2,0 \rangle$

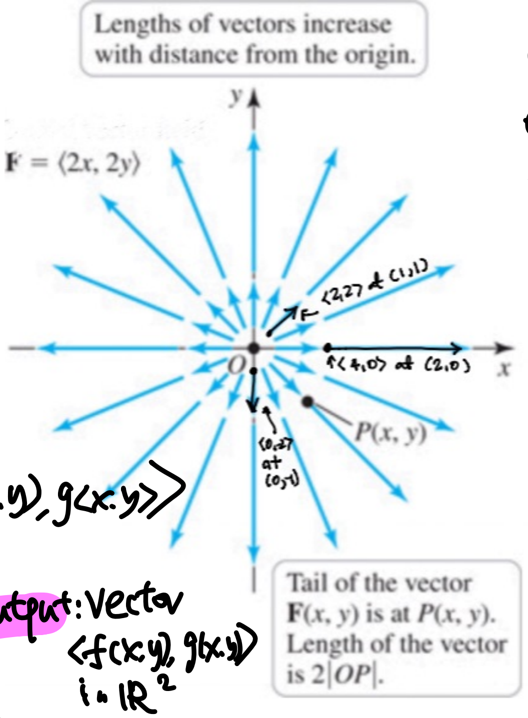
$F(-1,0) = \langle 2\cdot -1, 2\cdot 0 \rangle = \langle -2,0 \rangle$

$F(1,1) = \langle 2\cdot 1, 2\cdot 1 \rangle = \langle 2,2 \rangle$

$F(-1,1) = \langle 2\cdot -1, 2\cdot 1 \rangle = \langle -2,2 \rangle$

$F(-1,-1) = \langle 2\cdot -1, 2\cdot -1 \rangle = \langle -2,-2 \rangle$





once you plot enought number of vectors on $\mathbb{R}^2$. you get this graph! ←

but if you draw the actual vector, the graph gets cluttered. So dependig on the vector field, we scale the vectors.

Don't forget that $F(x,y) = \langle f(x,y), g(x,y) \rangle$ is a vector value function.

input → real number (x,y) in $\mathbb{R}^2$ (domain)

output: Vector $\langle f(x,y), g(x,y) \rangle$ in $\mathbb{R}^2$ (range).

17.2 → Line integral. integrate over in $\mathbb{R}^2$ & vector field in $\mathbb{R}^2$
(and $\mathbb{R}^3$ as well)] Exam2

17.3 + 17.4 → Sepecial case of line integral on a vector field.

17.6 → Surface integral integrate over a surface in $\mathbb{R}^3$ & vectorfield $\mathbb{R}^3$
: $\hat{n}$

17.7, 17.8 → Special case of surface integral on a vector field.

17.2 Line integrals.

- There are many ways to get parametric equation of different curves.

For straight lines, we can use

x_0 - first point
x_1 - last point

$\langle x, y\rangle = \langle x_0, y_0\rangle + t\langle x_1 - x_0, y_1 - y_0\rangle \quad 0 \le t \le 1.$

$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle \quad 0 \le t \le 1.$

For lines in $\mathbb{R}^2$, you can also use the equation of the line too.

ex) $y = 2x + 5 \qquad (-1, 3)$ to $(3, 11).$

$x = t \;, \quad y = 2t + 5 \quad -1 \le t \le 3.$

$x = t + 2 \;, \quad y = 2(t+2) + 5 \quad -1 \le t + 2 \le 3, \quad -3 \le t \le 1$

- For circle of radius r, center (a, b).

* counter clockwise orientation.

$x = a + r\cos t, \quad y = b + r\sin t \quad 0 \le t \le 2\pi$

works but there are many other variation of it.

- For general curve $f(x)=y$. let $x=t$. and $y=f(t)$. if $a \le x \le b$ (orientation from left to right). then $a \le t \le b$

ex). $y=(x+2)^2+3$ from $(-4,7)$ to $(-1,4)$.

let $x=t$ then $y=(t+2)^2+3$ $-4 \le t \le -1$.

$x=t-2$ then $y=(t-2+2)^2+3$ $-4 \le X \le -1$.

$$\downarrow$$

$= t^2+3$. $-4 \le t-2 \le -1$

$-2 \le t \le 1$

ex). $x=y^2-3$ from $(-2,1)$ to $(-3,0)$

let $y=t$ then $x=t^2-3$, $1 \le y \le 0 \to 1 \le t \le 0$.

※ with this method, the positive orientation is from left to right or from below to above.

with $\int_c f\,ds$, positive orientation of the curve does not matter

but $\int_c f \cdot \vec{n}\,ds$ (line integral on vector field it matters.

- For curve with orientation from right to left. and $a \le t \le b$

$x=t$, $y=f(t)$ ← oriented from left to right.

let $t(s) = (1-s)\,b + sa$ $s=1$ $t=a$ $s=0$ $t=b$.

and let $x=(1-s)\,b+sa$. $y = f(\underline{(1-s)\,b+sa})$, $0 \le s \le 1$.

replace t with this term.

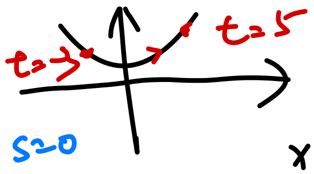
is the parametric eq. of the same curve with orientation from right to left.

ex). $f(x) = x^2+3$ from $(-3,12)$ to $(5,28)$.

let $x(t)=t$, $y(t)=t^2+3$, $-3 \le t \le 5$

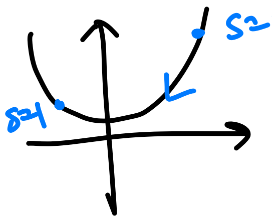
when $t=-3$, $x=-3$, $y=(-3)^2+3=12$ $(-3,12)$

when $t=5$, $x=5$, $y=(5)^2+3=28$ $(5,28)$



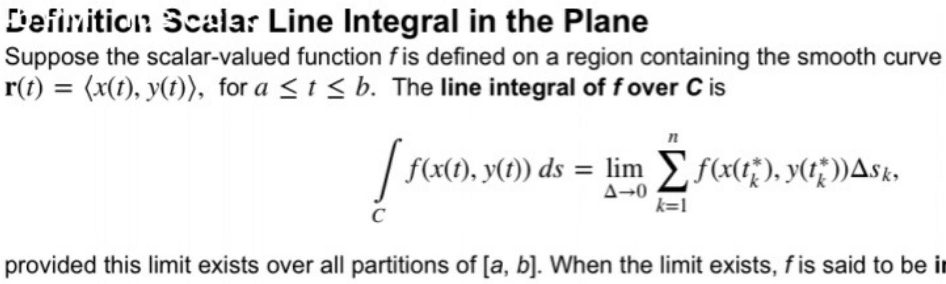
$t(s) = (1-s)(5)+s(3) = 5-5s-3s = 5-8s$.

$x=5-8s$. $y=f(5-8s) = (5-8s)^2+3$. $0 \le s \le 1$



when $s=0$, $x=5-8(0)=5$, $y=(5-8\cdot0)^2+3=28$ $(5,28)$

when $s=1$, $x=5-8(1)=-3$, $y=(5-8\cdot1)^2+3=12$ $(-3,12)$

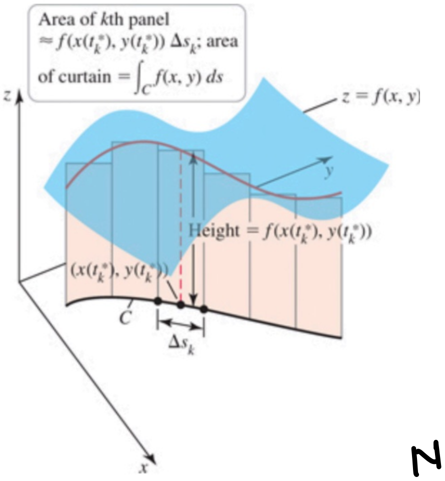


Definition Scalar Line Integral in the Plane

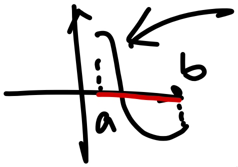
Suppose the scalar-valued function *f* is defined on a region containing the smooth curve *C* given by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$. The **line integral of *f* over *C*** is

$$\int_C f(x(t), y(t))\, ds = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x(t_k^*), y(t_k^*))\Delta s_k,$$

provided this limit exists over all partitions of [*a*, *b*]. When the limit exists, *f* is said to be **integrable** on *C*.



Integrating on a curve C is not too different from integrating f(x) with respect to X with $\int_a^b f(x)dx$, you are integrating on a straight line on x axis from a to b.



$\int_a^b f(x)dx$ is same as integrating on this real curve.

Now, we integrate f(x,y) on C, where C is a curve in $\mathbb{R}^3$. If $f(x,y) \geq 0$. $\int_C f ds$ gives area of a curtain like figure (rounded by C and projection of C on the surface of f(x,y)).

Theorem 17.1 Evaluating Scalar Line Integrals in $\mathbb{R}^2$

Let *f* be continuous on a region containing a smooth curve $C: \mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$. Then

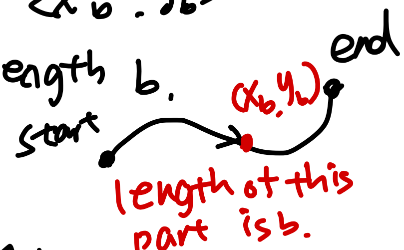
$$\int_C f\, ds = \int_a^b f(x(t), y(t)) |\mathbf{r}'(t)|\, dt$$

$$= \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

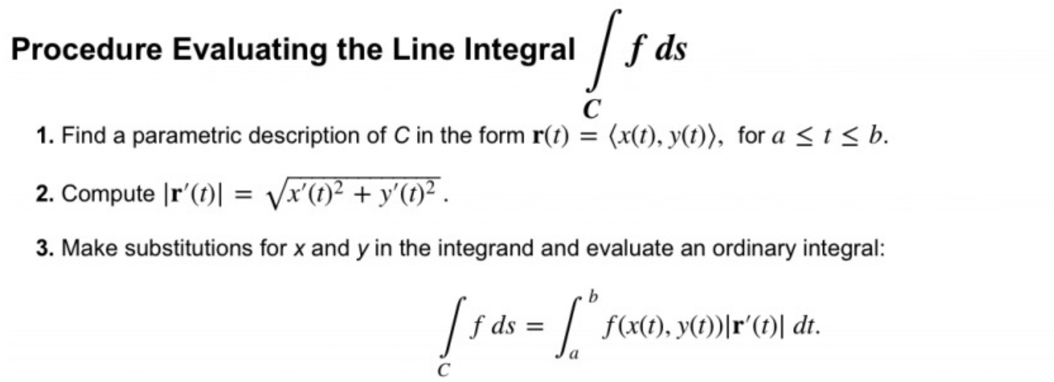
Q. Why is there a change of variable?

An. if a parametric. equation of curve C is given respect to its lenghth i.e. $r(s) = < x(s), y(s) >$. $0 \le s \le a$. where a is the Length of C. then $r(b) = < x(b). y(b) > : < x_b . y_b >$ you get coordinate which corresponds to the length b.

in that case $|r'(t)| = 1$ and $\int f ds = \int_0^a f(x(s), y(s)) ds$.



However, it is difficult to parametrize a curve with respect to its length. By multiplying $|r'(t)|$, we are changing the variable from s to t. Think of it as u-sub. Thus, $\int_C f ds = \int_C f(x(t), y(t)) |r'(t)| dt$



Procedure Evaluating the Line Integral $\int_C f\,ds$

1. Find a parametric description of C in the form $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, for $a \le t \le b$.

2. Compute $|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$.

3. Make substitutions for x and y in the integrand and evaluate an ordinary integral:

$$\int_C f\,ds = \int_a^b f(x(t), y(t))|\mathbf{r}'(t)|\,dt.$$

e.x). $\int_C 2x\,ds$; $C: r(t) = \langle \underset{\text{replace } x}{\underline{t}}, \underset{\text{replace } y}{\underline{t^2+1}} \rangle$ $\quad 0 \le t \le 2.$

1. Already done!

2. $r'(t) = \langle 1, 2t \rangle.$ $\quad |r'(t)| = \sqrt{1+(2t)^2} = \sqrt{1+4t^2}.$

3. $f(x(t), y(t)) = 2t$

$t=0, \quad u = 1+4\cdot 0^2 = 1.$

$t = 2. \quad u = 1+4(2)^2 = 17,$

$u = 1+4t^2, \; du = 8t\,dt \rightarrow \frac{1}{4}\,du = 2t\,dt.$

$$\int_C f\,ds = \int_0^2 2t\sqrt{1+4t^2}\,dt$$

$$= \int_1^{17} \sqrt{u}\,\frac{1}{4}\,du = \frac{1}{4}\,\frac{2}{3}\,u^{3/2}\Big|_1^{17} = \boxed{\frac{1}{6}(17)^{3/2} - \frac{1}{6}}.$$

e.g). $\int 4x^2 - 5y^2 ds$. c: part of line $y = 2x-3$ from $(-1,5)$ to $(3,3)$.

1. $r(t) = \langle \underset{\bar{x}}{t}, \underbrace{2t-3}_{y} \rangle$, $-1 \leq t \leq 3$.

2. $r'(t) = \langle 1. \ 2 \rangle$. $|r'(t)| = \sqrt{1^2+2^2} = \sqrt{5}$.

3. $f(x(t). y(t)) = 4t^2 - 5(2t-3)^2. = 4t^2 - 5(4t^2 - 12t + 9)$

$= 4t^2 - 20t^2 + 60t - 45$

$= -16t^2 + 60t - 45$.

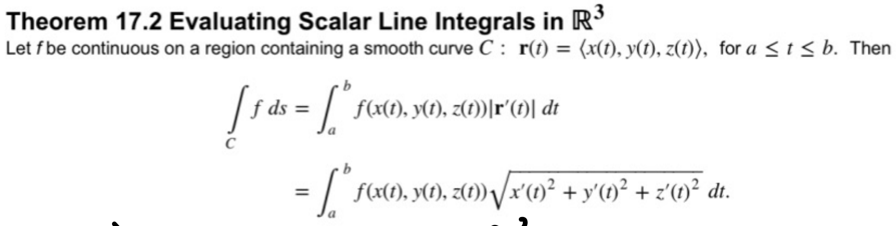
$$\int_c f ds = \int_{-1}^{3} (-16t^2 + 60t - 45)\sqrt{5}\, dt.$$

$$= \sqrt{5}\left(-\frac{16}{3}t^3 + 30t^2 - 45t\right)\Big|_{-1}^{2}$$

$$= \sqrt{5}\left[-\frac{16}{3}3^3 + 30(3)^2 - 45(3)) - \left(-\frac{16}{3}(-1)^3 + 30(-1)^2 - 45(-1)\right)\right]$$

$$= \sqrt{5}\left[(-144 + 270 - 135) - (-\frac{16}{3} + 30 + 45)\right]$$

$$\boxed{= \sqrt{5}\left(-9 - 75 + \frac{16}{3}\right).}$$



Theorem 17.2 Evaluating Scalar Line Integrals in $\mathbb{R}^3$

Let f be continuous on a region containing a smooth curve $C: \mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, for $a \le t \le b$. Then

$$\int_C f\, ds = \int_a^b f(x(t), y(t), z(t))|\mathbf{r}'(t)|\, dt$$

$$= \int_a^b f(x(t), y(t), z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt.$$

If $f(x,y,z)$ is a surface in $\mathbb{R}^3$ and C is a curve in $\mathbb{R}^3$.

$\int_C f\, ds$ is computed using the same method as $\mathbb{R}^2$ version.

e.g.) $\int_C xyz\, ds$ where $r(t) = \langle \underset{x}{\underline{\sin t}}, \underset{y}{\underline{\cos t}}, \underset{z.}{\underline{1}} \rangle$, $\frac{\pi}{2} \le t \le \pi$

1. Done!

2. $r'(t) = \langle \cos t, -\sin t, 0 \rangle$. $|r'(t)| = \sqrt{\cos^2 t + (-\sin^2 t) + 0} = \sqrt{\cos^2 t + \sin^2 t}$
$= 1.$

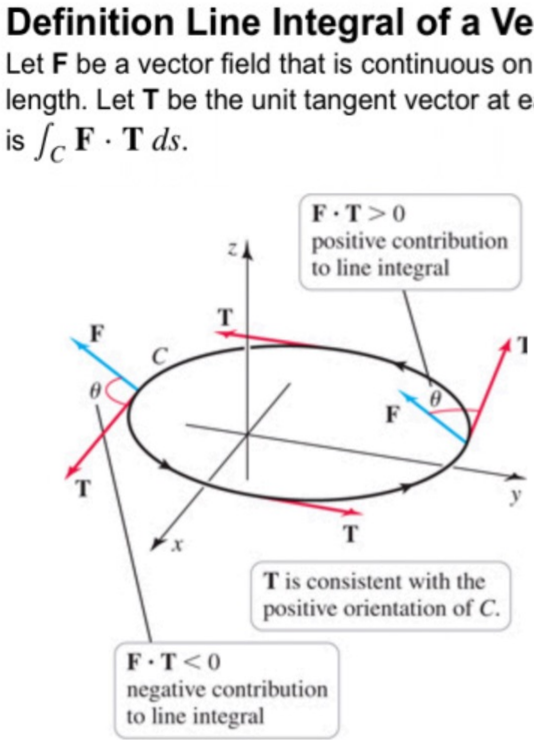
3. $f(x(t), y(t), z(t)) = \sin t \cdot \cos t \cdot 1$

$\int_C f\, ds = \int_{\frac{\pi}{2}}^{\pi} \sin t \cos t \cdot 1 \cdot 1\, dt$ let $u = \sin t$ $du = \cos t\, dt.$

$t = \pi$, $u = 0$
$t = \frac{\pi}{2}$, $u = 1$

$= \int_1^0 u\, du$

$= -\int_0^1 u\, du = -\frac{1}{2}u^2\Big|_0^1 = -\frac{1}{2}(1)^2 = -\frac{1}{2}.$



Definition Line Integral of a Vector Field

Let **F** be a vector field that is continuous on a region containing a smooth oriented curve *C* parameterized by arc length. Let **T** be the unit tangent vector at each point of *C* consistent with the orientation. The line integral of **F** over *C* is $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$.

$\mathbf{F} \cdot \mathbf{T} > 0$
positive contribution to line integral

T is consistent with the positive orientation of *C*.

$\mathbf{F} \cdot \mathbf{T} < 0$
negative contribution to line integral

$\vec{F}$: vector field (vector valued function)

$\vec{T}$: "unit" tangent vector at each point of C consistent with the orientation

In $\int_C \vec{F} \cdot \vec{T} ds$, we add dot product of $\vec{F}$ and $\vec{T}$ at each point on C.

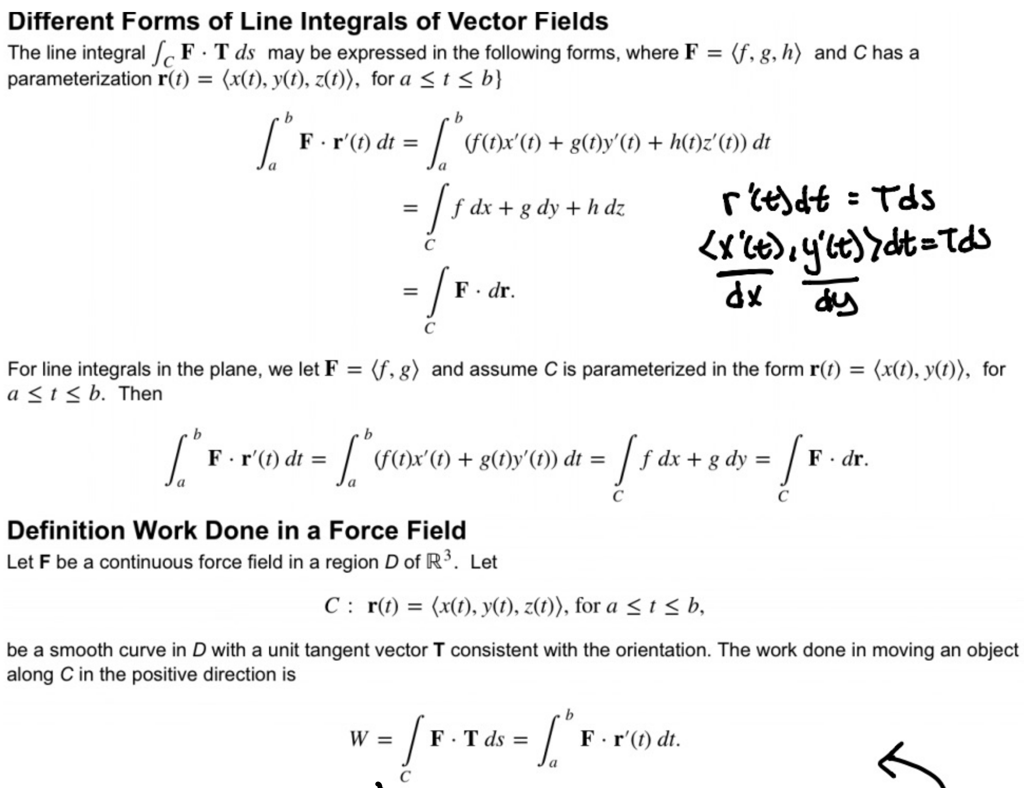
We can add them since $\vec{F} \cdot \vec{T}$ is a real number unlike $\int_C f ds$. orientation of the curve matters!

$\int_C \vec{F} \cdot \vec{T} ds$ – line integral of a vector field

$\vec{F} = \langle f(x, y), g(x, y) \rangle$ (a vector field in $\mathbb{R}^2$)

$\int_C f ds$ – scalar line integral

$f(x, y)$ (a function in $\mathbb{R}^2$)



Different Forms of Line Integrals of Vector Fields

The line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ may be expressed in the following forms, where $\mathbf{F} = \langle f, g, h \rangle$ and C has a parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, for $a \le t \le b\}$

$$\int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt = \int_a^b (f(t)x'(t) + g(t)y'(t) + h(t)z'(t))\, dt$$
$$= \int_C f\, dx + g\, dy + h\, dz$$
$$= \int_C \mathbf{F} \cdot d\mathbf{r}.$$

For line integrals in the plane, we let $\mathbf{F} = \langle f, g \rangle$ and assume C is parameterized in the form $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$. Then

$$\int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt = \int_a^b (f(t)x'(t) + g(t)y'(t))\, dt = \int_C f\, dx + g\, dy = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Definition Work Done in a Force Field

Let **F** be a continuous force field in a region D of $\mathbb{R}^3$. Let

$$C: \ \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, \text{ for } a \le t \le b,$$

be a smooth curve in D with a unit tangent vector **T** consistent with the orientation. The work done in moving an object along C in the positive direction is

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt.$$

$$r'(t)dt = Tds$$
$$\langle \underbrace{x'(t)}_{dx}, \underbrace{y'(t)}_{dy} \rangle dt = Tds$$

Basic idea:

① Replace x, y, z in $\vec{F} = \langle f, g, h \rangle$ with $x(t), y(t), z(t)$ in $r(t)$

② compute $r'(t)$

③ compute $\vec{F} \cdot r'(t)$

$$\vec{F} \cdot r'(t)$$
$$= fx' + gy' + hz'$$
$$\text{or} = f\underline{dx} + gdy + hdz.$$

derivative of x

in physics, $W = \vec{F}S$ is a very simple example of $\vec{F}$ is a constant force field (meaning $\vec{F}$ is same at all points) s is a distance in a straight line.

e.g.) $\int_C \vec{F}\cdot\vec{T}\,ds$ $\quad \vec{F} = \langle x-y, x\rangle.$ $\quad$ C: line segment from $(-1,2)$ to $(0,5)$

① $C: y-5 = \frac{3}{1}(x-1)$

$y = 3x+5$ $\quad \longrightarrow \quad r(t) = \langle \underset{\substack{x \\ \text{component}}}{\underline{t}}, \underset{\substack{y \\ \text{component}}}{\underline{3t+5}} \rangle. \qquad -1 \le t \le 0.$

② $r'(t) = \langle \underset{\substack{x'(t) \\ dx}}{\underline{1}}, \underset{\substack{y'(t) \\ dy}}{\underline{3}} \rangle$

③ $\vec{F} = \langle t-(3t+5), t\rangle = \langle \underset{f}{\underline{-2t-5}}, \underset{g}{\underline{t}} \rangle$

④ $\vec{F}\cdot\vec{T} = \vec{F}\cdot r'(t) = \langle f(t), g(t)\rangle \circ \langle x'(t), y'(t)\rangle$

$= f\,x'(t) + g\,y'(t) = f\,dx + g\,dy$

$= (-2t-5)(1) + 3(t) = -2t-5+3t = t-5$

⑤ $\int_C \vec{F}\cdot\vec{T}\,ds = \int_{-1}^{0} t-5\,dt = \frac{1}{2}t^2 - 5t \Big|_{-1}^{0} = (\frac{1}{2}\cdot 0^2 - 5\cdot 0) - (\frac{1}{2}(-1)^2 - 5(-1))$

$= 0 - (\frac{1}{2} + 5) = -\frac{1}{2}.$

Definition Circulation

Let **F** be a continuous vector field on a region *D* of $\mathbb{R}^3$, and let *C* be a closed smooth oriented curve in *D*. The **circulation** of **F** on *C* is $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$, where **T** is the unit vector tangent to *C* consistent with the orientation.

Circulation We assume $\mathbf{F} = \langle f, g, h \rangle$ is a continuous vector field on a region *D* of $\mathbb{R}^3$, and we take *C* to be a *closed* smooth oriented curve in *D*. The *circulation* of **F** along *C* is a measure of how much of the vector field points in the direction of *C*. More simply, as you travel along *C* in the positive direction, how much of the vector field is at your back and how much of it is in your face? To determine the circulation, we simply "add up" the components of **F** in the direction of the unit tangent vector **T** at each point. Therefore, circulation integrals are another example of line integrals of vector fields.

To compute circulation, you simply compute line integral on vectorfield.

But a circulation of $\vec{F}$ on C only applies to a closed curve.

ex). compute circulation on $C: r(t)=\langle \underset{x}{\sin(t)}, \underset{y}{\cos(t)} \rangle$ $0 \le t \le 2\pi$

of the vector field $F = \langle \underset{f}{x}, \underset{g}{-y} \rangle$

$$\int f\,dx + g\,dy$$

$$= \int_0^{2\pi} \langle \sin t, -\cos t \rangle \cdot \langle \cos t, -\sin t \rangle \, dt$$

$$= \int_0^{2\pi} \sin t \cos t + \sin t \cos t \, dt$$

$$= \int_0^{2\pi} 2 \sin t \cos t \, dt = \int_0^0 2u \, du = 0$$

$u = \sin t$ $du = \cos t\,dt$

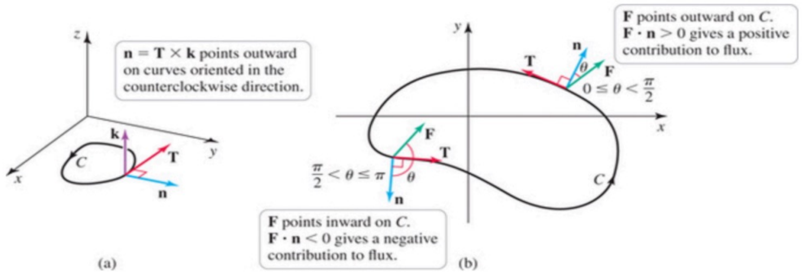
$t = 2\pi$, $u = \sin 2\pi = 0$

$t = 0$, $u = \sin 0 = 0$.

Flux of Two-Dimensional Vector Fields Assume $\mathbf{F} = \langle f, g \rangle$ is a continuous vector field on a region *R* of $\mathbb{R}^2$. We let *C* be a smooth oriented curve in *R* that does not intersect itself; *C* may or may not be closed. To compute the *flux* of the vector field across *C*, we "add up" the components of **F** *orthogonal* or *normal* to *C* at each point of *C*. Notice that every point on *C* has two unit vectors normal to *C*. Therefore, we let **n** denote the unit vector in the *xy*-plane normal to *C* in a direction to be defined momentarily. Once the direction of **n** is defined, the component of **F** normal to *C* is $\mathbf{F} \cdot \mathbf{n}$, and the flux is the line integral of $\mathbf{F} \cdot \mathbf{n}$ along *C*, which we denote $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$.

The first step is to define the unit normal vector at a point *P* of *C*. Because *C* lies in the *xy*-plane, the unit vector **T** tangent at *P* also lies in the *xy*-plane. Therefore, its *z*-component is 0, and we let $\mathbf{T} = \langle T_x, T_y, 0 \rangle$. As always, $\mathbf{k} = \langle 0, 0, 1 \rangle$ is the unit vector in the *z*-direction. Because a unit vector **n** in the *xy*-plane normal to *C* is orthogonal to both **T** and **k**, we determine the direction of **n** by letting $\mathbf{n} = \mathbf{T} \times \mathbf{k}$. This choice has two implications.

- If *C* is a closed curve oriented counterclockwise (when viewed from above), the unit normal vector points *outward* along the curve (**Figure 17.26a**). When **F** also points outward at a point on *C*, the angle θ between **F** and **n** satisfies $0 \le \theta < \frac{\pi}{2}$ (**Figure 17.26b**). At all such points, $\mathbf{F} \cdot \mathbf{n} > 0$ and there is a positive contribution to the flux across *C*. When **F** points inward at a point on *C*, $\frac{\pi}{2} < \theta \le \pi$ and $\mathbf{F} \cdot \mathbf{n} < 0$, which means there is a negative contribution to the flux at that point.



d

Figure 17.26

- If *C* is not a closed curve, the unit normal vector points to the right (when viewed from above) as the curve is traversed in the positive direction.

Definition Flux

Let $\mathbf{F} = \langle f, g \rangle$ be a continuous vector field on a region *R* of $\mathbb{R}^2$. Let $C: \mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$, be a smooth oriented curve in *R* that does not intersect itself. The **flux** of the vector field **F** across *C* is

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b (f(t)y'(t) - g(t)x'(t))\, dt,$$

where $\mathbf{n} = \mathbf{T} \times \mathbf{k}$ is the unit normal vector and **T** is the unit tangent vector consistent with the orientation. If *C* is a closed curve with counterclockwise orientation, **n** is the outward normal vector, and the flux integral gives the **outward flux** across *C*.

The concepts of circulation and flux can be visualized in terms of headwinds and crosswinds. Suppose the wind patterns in your neighborhood can be modeled with a vector field **F** (that doesn't change with time). Now imagine taking a walk around the block in a counterclockwise direction along a closed path. At different points along your walk, you encounter winds from various directions and with various speeds. The circulation of the wind field **F** along your path is the net amount of headwind (negative contribution) and tailwind (positive contribution) that you encounter during your walk. The flux of **F** across your path is the net amount of crosswind (positive from your left and negative from your right) encountered on your walk.

unlike circulation, to flux of the vector field. across C applies to both closed & not closed curves

actually both $\langle -y', x' \rangle$ $\langle -y', -x' \rangle$ are orthogonal to unit tengent vector T.

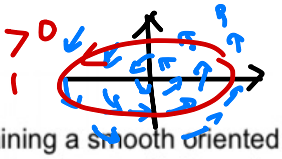
However, when we fix the vector K which is orthogonal to both $\vec{T}$ and $\vec{n}$, to be $\langle 0.0.1 \rangle$ instead of $\langle 0.0.-1 \rangle$. we choose $\langle y', -x' \rangle$ to be used in the definition of flux.

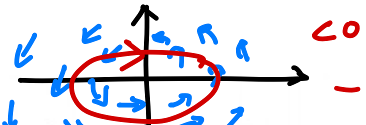
ex). Compute flux on $C: r(t) = \langle \underset{x}{\sin t}, \underset{y}{\cos t} \rangle$ $0 \le t \le 2\pi$ of the vector field $\vec{F} = \langle \underset{f}{x}, \underset{g}{-y} \rangle$

$$\int_0^{2\pi} f\,dy - g\,dx = \int_0^{2\pi} \sin t(-\sin t) - (-\cos t)(\cos t)\,dt$$

$$= \int_0^{2\pi} -\sin^2 t + \cos^2 t \; dt$$

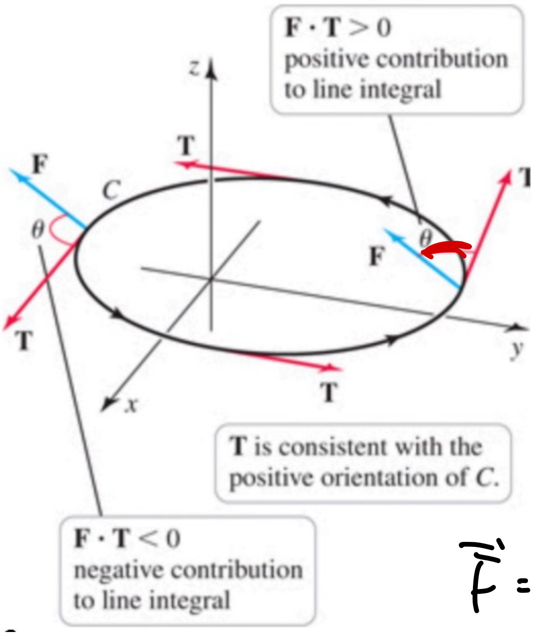
$$= \int_0^{2\pi} \cos 2t\,dt = \frac{1}{2}\sin 2t \Big|_0^{2\pi} = 0.$$





Definition Line Integral of a Vector Field

Let **F** be a vector field that is continuous on a region containing a smooth oriented curve *C* parameterized by arc length. Let **T** be the unit tangent vector at each point of *C* consistent with the orientation. The line integral of **F** over *C* is $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$.



$\vec{F}$: vector Field. (vector valued functn

$\vec{T}$: "unit" tangent. vector at each point of C.

$|\vec{T}|=1$. consistent with the orientation.

$\int_C \vec{F}\cdot\vec{T}\, ds$. "dot product" of $\vec{F}$ and $\vec{T}$. at each point on C.

$\vec{F}\cdot\vec{T} = |\vec{F}|\,|\vec{T}|\cos\theta = |\vec{F}|\cos\theta$.

unlike $\int_C f ds$, orientation of curve matters.

$\vec{F} = \langle f(x,y), g(x,y)\rangle$. (a vector field in $\mathbb{R}^2$)

$\int_C f\, ds$ — Scalar line integral $f(x,y)$ (a func in $\mathbb{R}^2$).

Different Forms of Line Integrals of Vector Fields

The line integral $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ ~~may be~~ expressed in the following forms, where $\mathbf{F} = \langle f, g, h\rangle$ and C has a parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, for $a \le t \le b\}$

$$\int_C \vec{F}\cdot\vec{T}\,\underbrace{ds}_{|\vec{r}'(t)|dt} = \int_C \vec{F}\cdot\frac{\vec{r}'(t)}{\cancel{|\vec{r}'(t)|}}\cancel{|\vec{r}'(t)|}\,dt = \int_C \vec{F}\cdot\vec{r}'(t)\,dt$$

$$\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_a^b (f(t)x'(t) + g(t)y'(t) + h(t)z'(t))\,dt$$

$$= \int_C f\,dx + g\,dy + h\,dz$$

$$= \int_C \mathbf{F}\cdot d\mathbf{r}.$$

$$r'(t)dt = Tds$$

$$\langle \underbrace{x'(t)}_{dx}, \underbrace{y'(t)}_{dy}\rangle dt = Tds$$

Basic idea:

① Replace x, y, z in $\vec{F} \langle f, g, h\rangle$. with $x(t)$. $y(t)$. $z(t)$ in $\vec{r}(t)$.

② $\vec{r}'(t)$ compute.

③ $\vec{F}\cdot\vec{r}'(t)$.

$$= \langle f, g, h\rangle\cdot\langle x'(t), y'(t), z'(t)\rangle$$

$$= fx' + gy' + hz'.$$

$x'(t)dt = dx$. $y'(t)dt = dy$ $z'(t)dt = dz$

$$= fdx + gdy + hdz.$$

For line integrals in the plane, we let $\mathbf{F} = \langle f, g\rangle$ and assume C is parameterized in the form $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, for $a \le t \le b$. Then

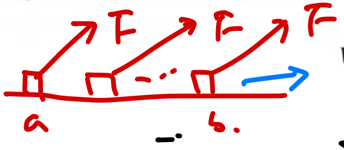
$$\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_a^b (f(t)x'(t) + g(t)y'(t))\,dt = \int_C f\,dx + g\,dy = \int_C \mathbf{F}\cdot d\mathbf{r}.$$

Definition Work Done in a Force Field

Let **F** be a continuous force field in a region D of $\mathbb{R}^3$. Let

$$C:\ \mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle,\ \text{for } a \le t \le b,$$

be a smooth curve in D with a unit tangent vector **T** consistent with the orientation. The work done in moving an object along C in the positive direction is

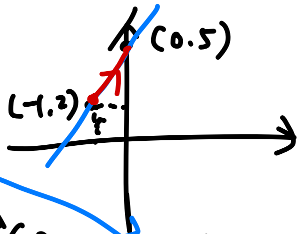


$$W = \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt.$$

$W = \vec{F}\cdot s$ in physics. simple example $\vec{F}$ is a constant force field.

s is a distance in a straight line.

e.g). $\int_C \vec{F}\cdot \vec{T}\,ds$. $\vec{F} = \langle x-y, x\rangle$ C: line segment from (−1, 2) to (0.5).



① $y = mx + 5$.

$2 = m(-1) + 5$.

$m = \frac{2-5}{-1} = \frac{-3}{-1} = 3$.

$y = 3x + 5$.

$x = t$

$y = 3t + 5$. $\vec{r}(t) = \langle t, 3t+5\rangle$.

② $\vec{r}\,'(t) = \langle 1, 3\rangle$.

$\boxed{\bar{x}'(t)}$ $\boxed{\bar{y}'(t)}$

$-1 \le x \le 0$.

$\boxed{-1 \le t \le 0}$

③ $\vec{F} = \langle t-(3t+5), t\rangle = \langle \underbrace{-2t-5}_{f}, \underbrace{t}_{g}\rangle$.

④ $\vec{F}\cdot\vec{r} = \langle -2t-5, t\rangle \cdot \langle 1, 3\rangle = -2t - 5 + 3t = \boxed{t-5}$

$\langle f(t), g(t)\rangle \cdot \langle x'(t), y'(t)\rangle = f(t)x'(t) + g(t)y'(t)$

⑤ $\int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}\cdot\vec{r}\,dt = \int_{-1}^{0} t-5\,dt = \frac{1}{2}t^2 - 5t \Big|_{-1}^{0} = -\left[\frac{1}{2}(-1)^2 - 5(-1)\right]$

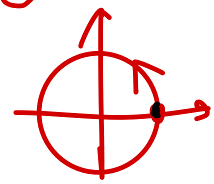
$= -\left[\frac{1}{2} + 5\right] = -\frac{11}{2}$

Definition Circulation

Let **F** be a continuous vector field on a region *D* of $\mathbb{R}^3$, and let *C* be a closed smooth oriented curve in *D*. The **circulation** of **F** on *C* is $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$, where **T** is the unit vector tangent to *C* consistent with the orientation.

Circulation We assume $\mathbf{F} = \langle f, g, h \rangle$ is a continuous vector field on a region *D* of $\mathbb{R}^3$, and we take *C* to be a *closed* smooth oriented curve in *D*. The *circulation* of **F** along *C* is a measure of how much of the vector field points in the direction of *C*. More simply, as you travel along *C* in the positive direction, how much of the vector field is at your back and how much of it is in your face? To determine the circulation, we simply "add up" the components of **F** in the direction of the unit tangent vector **T** at each point. Therefore, circulation integrals are another example of line integrals of vector fields.

C: closed. To compute circulation, you simply complete line integral over vectorfield. But only applies to a closed curve

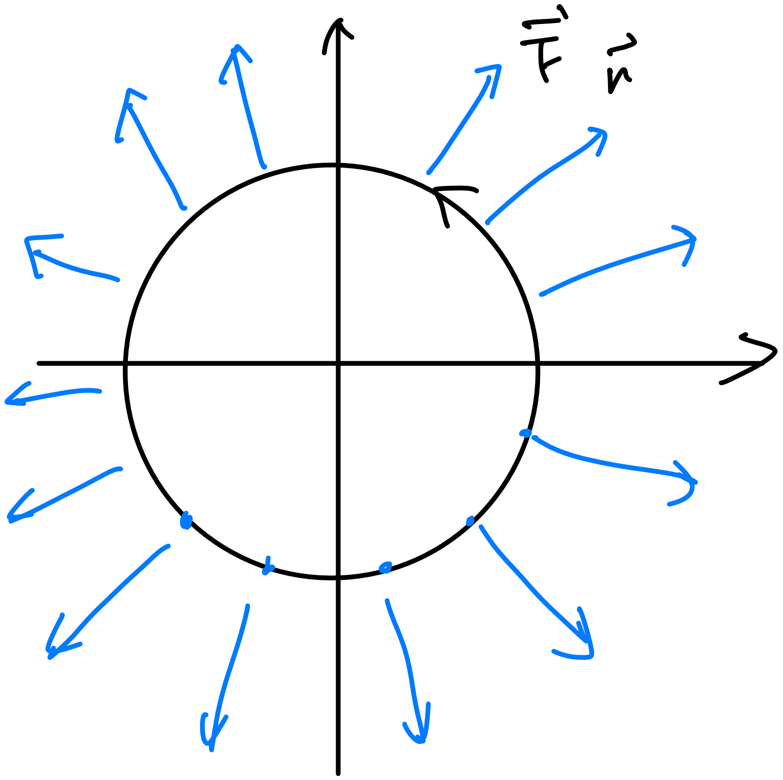


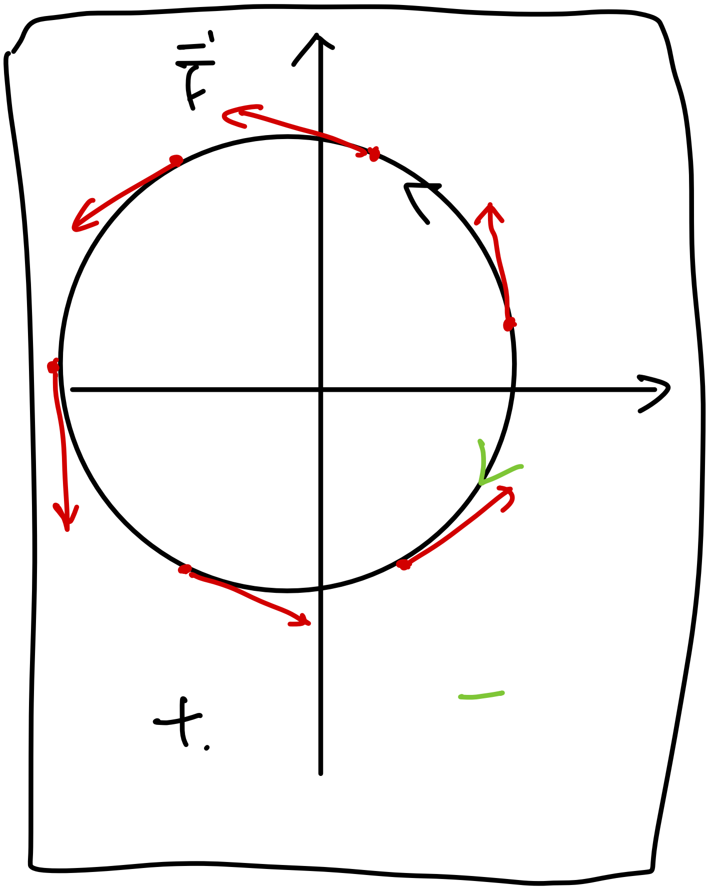
e.g.) compute circulation on $C: \vec{r}(t) = \langle \cos t, \sin t \rangle,\ 0 \le t \le 2\pi$ of the vector field $F = \langle x, -y \rangle = \langle \cos t, -\overset{x}{\sin} t \rangle\ \overset{}{y}.$ $\vec{r}'(t) = \langle -\sin t, \cos t \rangle$

$$\int \vec{F} \cdot \vec{r}' dt = \int_0^{2\pi} \underset{f \,\cdot\, g}{\langle \cos t, -\sin t \rangle \cdot \langle -\sin t, \cos t \rangle}\, dt.$$

$u = \sin t \quad du = \cos t\, dt$

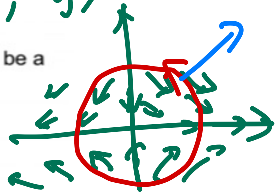
$$= \int_0^{2\pi} -\sin t \cos t - \sin t \cos t\, dt = \int_0^{2\pi} -2 \sin t \cos t\, dt = \int -2u\, du = -u^2 \Big|_0^{2\pi} = -\sin^2 t \Big|_0^{2\pi} = 0$$





Circulati	0.	
Flux.	+	0.

$\vec{F} = \langle x, -y \rangle$



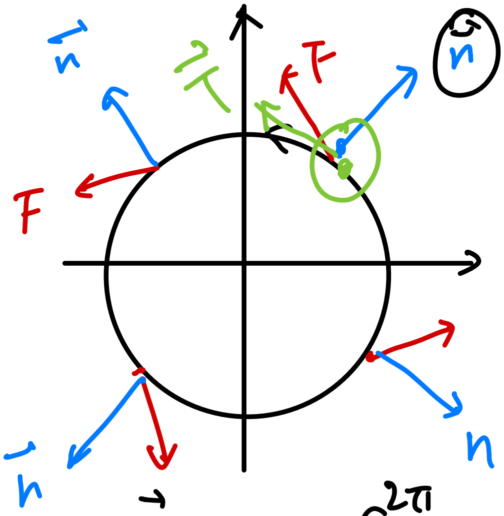
Definition Flux

Let $\mathbf{F} = \langle f, g \rangle$ be a continuous vector field on a region R of $\mathbb{R}^2$. Let $C:\ \mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$, be a smooth oriented curve in R that does not intersect itself. The **flux** of the vector field **F** across C is

$$\int_C \mathbf{F} \cdot \boxed{\mathbf{n}}\, ds = \boxed{\int_a^b (\underbrace{f(t)y'(t)}_{\text{I}} - \underbrace{g(t)x'(t)}_{\text{II}})\, dt,}$$

where $\boxed{\mathbf{n} = \mathbf{T} \times \mathbf{k}}$ is the unit normal vector and **T** is the unit tangent vector consistent with the orientation. If C is a closed curve with counterclockwise orientation, **n** is the outward normal vector, and the flux integral gives the **outward flux** across C.

$\boxed{\langle y', -x' \rangle}$



$\vec{T} : \vec{r}\,'(t) = \langle x', y' \rangle.$ $\nearrow$ ~~$\langle -y', x' \rangle$~~

$\searrow$ $\hat{n} = \underline{\langle y', -x' \rangle}.$

e

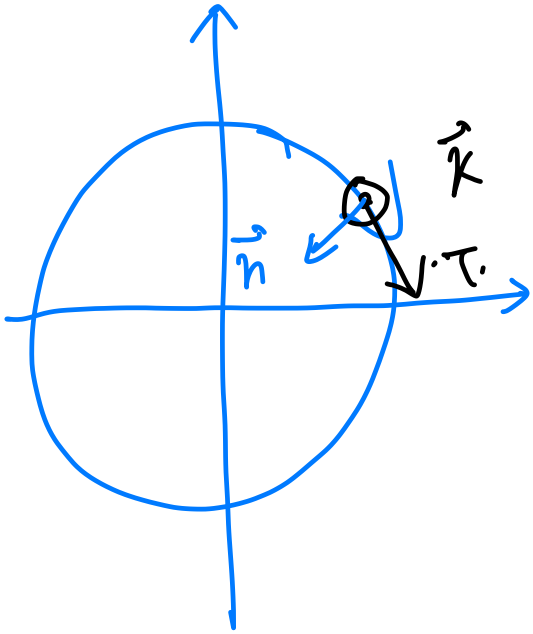
ex) Complut flux on $C : \vec{r}(t) = \langle \overset{x}{\cos t}, \overset{y}{\sin t} \rangle.$

$0 \le t \le 2\pi$

$\vec{F} = \langle \underset{f}{x}, \underset{g}{-y} \rangle.$ y' $\boxed{\langle y', -x' \rangle} = \langle \cos t, \sin t \rangle.$

$$\int \vec{F} \cdot \vec{n}\, ds = \int_0^{2\pi} \langle \cos t, -\sin t \rangle \cdot \boxed{\langle \cos t, \sin t \rangle}\, dt.$$

$$= \int_0^{2\pi} \underline{\cos^2 t - \sin^2 t}.\ dt = \int_0^{2\pi} \underline{\cos 2t dt} = \underline{\tfrac{1}{2}\sin 2t}\Big|_0^{2\pi} = 0.$$



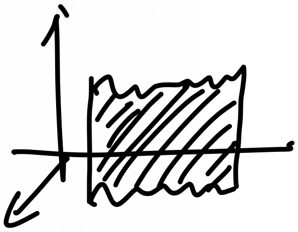
$\vec{k}$
$\vec{n}$
$\vec{\tau}$

Summary

- Scalar Line Integral: (Given scalar valued function f and parameterized curve C.)

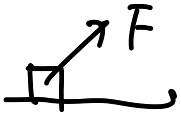
$$\int_C f\,ds = \int_a^b f\big(x(t),y(t)\big)|\mathbf{r}'(t)|\,dt$$

Application: Area of the "curtain" between f and C, when $f > 0$.



- Line Integral of a Vector Field: (Given vector field $\mathbf{F}$ and parameterized curve C.)

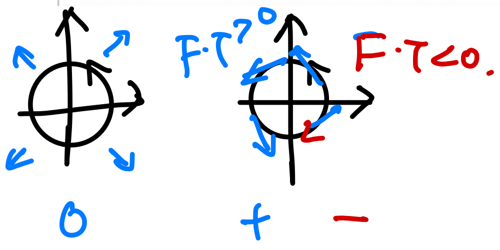
$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}\cdot\underline{\mathbf{r}'(t)}\,dt = \int_a^b \langle f, g\rangle \cdot \underline{\langle x', y'\rangle}\,dt.$$

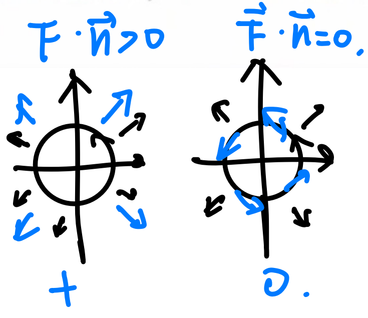


Application: Work done by the force field $\mathbf{F}$ in moving an object along the curve C.

- Circulation: $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ (for closed C) Flux: $\int_C \mathbf{F}\cdot\underline{\mathbf{n}}\,ds$

$$= \int_a^b \langle f, g\rangle \langle y', -x'\rangle\,dt.$$





17.3. Conservative vector fields.

Definition Conservative Vector Field

A vector field **F** is said to be **conservative** on a region (in $\mathbb{R}^2$ or $\mathbb{R}^3$) if there exists a scalar function φ such that $\mathbf{F} = \nabla\varphi$ on that region.

The term *conservative* refers to conservation of energy. See Exercise 66 for an example of conservation of energy in a conservative force field.

Theorem 17.3 Test for Conservative Vector Fields

Let $\mathbf{F} = \langle f, g, h \rangle$ be a vector field defined on a connected and simply connected region *D* of $\mathbb{R}^3$, where *f*, *g*, and *h* have continuous first partial derivatives on *D*. Then **F** is a conservative vector field on *D* (there is a potential function φ such that $\mathbf{F} = \nabla\varphi$) if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \text{and} \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}.$$

For vector fields in $\mathbb{R}^2$, we have the single condition $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

Procedure Finding Potential Functions in $\mathbb{R}^3$

Suppose $\mathbf{F} = \langle f, g, h \rangle$ is a conservative vector field. To find φ such that $\mathbf{F} = \nabla\varphi$, use the following steps:

1. Integrate $\varphi_x = f$ with respect to *x* to obtain φ, which includes an arbitrary function *c*(*y*, *z*).
2. Compute φ_y and equate it to *g* to obtain an expression for $c_y(y, z)$.
3. Integrate $c_y(y, z)$ with respect to *y* to obtain *c*(*y*, *z*), including an arbitrary function *d*(*z*).
4. Compute φ_z and equate it to *h* to get *d*(*z*).

A similar procedure beginning with $\varphi_y = g$ or $\varphi_z = h$ may be easier in some cases.

ex) $F = \langle 2x, 3y^2 \rangle$

$f(x \cdot y) = x^2 + y^3 + 5$

$\nabla f(x \cdot y) = \langle f_x, f_y \rangle$

$= \langle 2x, 3y^2 \rangle$

$= \vec{F}$

f is the function φ in the definition. It is also called a potential function.

Using Thm 17.3. we check if $\vec{F}$ is a conservative vector field.

Yes → Find potential fun

No → done

ex). $\vec{F} = \langle e^x \cos y,\ -e^x \sin y \rangle$.

f g.

$f_y = -e^x \sin y$ $g_x = -e^x \sin y$ $f_y = g_x$ so F is conservative

ex) $\vec{F} = \langle 2xy - z^2,\ x^2 + 2z,\ 2y + 2xz \rangle$.

$f_y = 2x$ $g_x = 2x$, $f_z = -2z$ $h_x = 2z$

$f_y = g_x$ $f_z \neq h_x$ $\rightarrow$ since $h_x \neq f_z$

F is not conservative.

ex) $\vec{F} = \langle 2xy - z^2,\ x^2 + 2z,\ 2y - 2xz \rangle$ $\rightarrow$ F is conservative.

$f_y = 2x$ $g_x = 2x$, $f_z = -2z$, $h_x = -2z$, $g_z = 2$. $h_y = 2$

$f_y = g_x$ $f_z = h_x$ $g_z = h_y$

ex) Find potential function for $\vec{F} = \langle e^x \cos y, -e^x \sin y \rangle$

we are trying to find a function^{f} φ such $\overset{g}{\text{that}}$

$\varphi_x = e^x \cos y$ and $\varphi_y = -e^x \sin y$

Since $\varphi_x = e^x \cos y$

$$\int \varphi_x dx = \int e^x \cos y \, dx = \varphi$$

$$= e^x \cos y + C(y)$$

so $\varphi = e^x \cos y + C(y)$

Since we also have variable y when we integrate with respect to x, we don't just have +c but C(y). a function which contains y and constants.

we need to figure out what C(y) is.

Since $\varphi = e^x \cos y + C(y)$

$\varphi_y = -e^x \sin y + C'(y)$. But $\varphi_y = -e^x \sin y$ as well.

Then $C'(y) = 0$

Thus $C(y) = \int C'(y) dy = \int 0 dy = C$, C is a constant.

so $\varphi = e^x \cos y + C$.

ex) find potential function for $\vec{F} = \langle \underbrace{2xy - z^2}_{f}, \underbrace{x^2 + 2z}_{g}, \underbrace{2y - 2xz}_{h} \rangle$

Again, find φ such that $\varphi_x = f$, $\varphi_y = g$ and $\varphi_z = h$.

since $\varphi_z = h$

$\varphi = \int \varphi_z \, dz = \int 2y - 2xz \, dz = 2yz - xz^2 + \underset{*}{\underline{\underline{C(x,y)}}}$

* function contains x and y as well so when we integrate w.r. to z, we get a function with x and y.

$\varphi = 2yz - xz^2 + C(x,y)$

$\varphi_x = -z^2 + C_x(x,y) = f = 2xy - z^2$

Then $C_x(x,y) = 2xy$

so, $C(x,y) = \int C_x(x,y) \, dx = \int 2xy \, dx = x^2y + \underset{**}{\underline{\underline{C(y)}}}$

** we know everything for variable z thus we only get C(y)

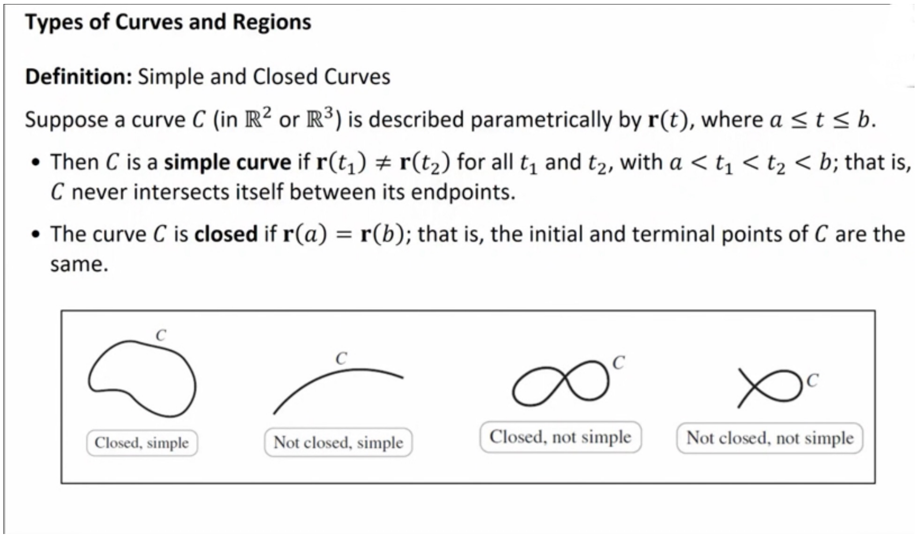
Then, $\varphi = 2yz - xz^2 + x^2y + C(y)$

$\varphi_y = 2z + C'(y) = g = x^2 + 2z$

Then $C'(y) = x^2$.

so $C(y) = \int C'(y) dy = \int x^2 dy = x^2y + c$

so $\varphi = 2yz - xz^2 + x^2y + C$.

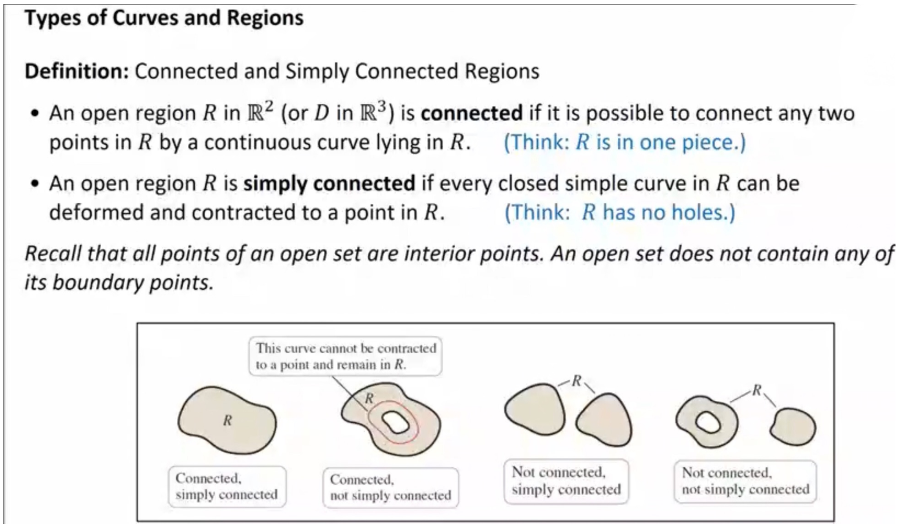


Types of Curves and Regions

Definition: Simple and Closed Curves

Suppose a curve C (in $\mathbb{R}^2$ or $\mathbb{R}^3$) is described parametrically by $\mathbf{r}(t)$, where $a \le t \le b$.

- Then C is a **simple curve** if $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ for all t_1 and t_2, with $a < t_1 < t_2 < b$; that is, C never intersects itself between its endpoints.
- The curve C is **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$; that is, the initial and terminal points of C are the same.



Types of Curves and Regions

Definition: Connected and Simply Connected Regions

- An open region R in $\mathbb{R}^2$ (or D in $\mathbb{R}^3$) is **connected** if it is possible to connect any two points in R by a continuous curve lying in R. (Think: R is in one piece.)
- An open region R is **simply connected** if every closed simple curve in R can be deformed and contracted to a point in R. (Think: R has no holes.)

Recall that all points of an open set are interior points. An open set does not contain any of its boundary points.

Theorem 17.4 Fundamental Theorem for Line Integrals

Let R be a region in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let φ be a differentiable potential function defined on R. If $\mathbf{F} = \nabla\varphi$ (which means that $\mathbf{F}$ is conservative), then

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A),$$

for all points A and B in R and all piecewise-smooth oriented curves C in R from A to B.

A and B are coordinates of the first & last pts of the curve C.

Compare the two versions of the Fundamental Theorem.

$$\int_a^b F'(x)\, dx = F(b) - F(a) \leftarrow \text{Calc2}$$

$$\int_C \nabla\varphi \cdot d\mathbf{r} = \varphi(B) - \varphi(A) \leftarrow \text{Calc4}$$

For line integral, we get Fund. thm if and only if when $\vec{F}$ is conservative.

Theorem 17.5

Let $\mathbf{F}$ be a continuous vector field on an open connected region R in $\mathbb{R}^2$. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, then $\mathbf{F}$ is conservative; that is, there exists a potential function φ such that $\mathbf{F} = \nabla\varphi$ on R.

Thm 17.5 is an another way to determine whether or not $\vec{F}$ is conservative vector field. It means if $\int_C \vec{F} \cdot dr$ is same for <u>all</u> types of curves, ~~then~~ $\vec{F}$ is conservative. But this is not possible to test as you can't check for all curves. :|

ex). $\int_C \vec{F}\cdot dr$, $\vec{F} = \langle 2xy - z^2, x^2+2z, 2y-2xz \rangle$

$C: r(t) = \langle t^2+1, e^{t}+1, t+3 \rangle$, $-3 \le t \le 0$.

since $\vec{F}$ is conservative, by FTL.

$\int_C \vec{F}\cdot dr = \varphi(B) - \varphi(A)$ where φ is a potential function and B, A the Last & first pt of the curve C.

$\varphi(x.y.z) = 2yz - xz^2 + x^2y + c.$

first pt: $r(-3) = \langle 10, e^{-3}+1, 0 \rangle \rightarrow (10, e^{-3}+1.0)$

Last pt: $r(0) = \langle 1.2.3 \rangle \rightarrow (1.2.3\rangle$

so. $\int_C \vec{F}\cdot dr = \varphi(1,2,3) - \varphi(10, e^{-3}+1, 0)$

$= (2\cdot 2\cdot 3 - 1\cdot 3^2 + 1^2\cdot 2 + c) - (2(e^{-3}+1)\cdot 0 - 10\cdot 0^2 + 10^2(e^{-3}+1) + c)$

$= (12 - 9 + 2 + c) - (10^2(e^{-3}+1) + c)$

$= 5 - 10^2(e^{-3}+1)$

* see how c are cancelled? You can drop c from φ when computing line integral using FTL.

ex. $\varphi(x,y,z) = xy + xz + yz$. ← potential function. $\vec{F} = \nabla\varphi$.

C: $r(t) = \langle \underbrace{t, 2t, 3t}_{\text{Line segment.}} \rangle$. $0 \le t \le 4$.

$x = t,\ y = 2t,\ z = 3t.$

a. $\boxed{\text{use } \int_a^b \vec{F}\cdot r'(t)\,dt.}$

$\vec{F} = \nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z \rangle = \langle y+z, x+z, x+y \rangle.$

$\vec{F}(\vec{r}(t)) = \langle 2t+3t, t+3t, t+2t \rangle = \langle 5t, 4t, 3t \rangle.$

$\vec{r}'(t) = \frac{d}{dt}\langle t, 2t, 3t \rangle = \langle 1, 2, 3 \rangle.$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = \langle 5t, 4t, 3t \rangle \cdot \langle 1, 2, 3 \rangle$

$= 5t + 8t + 9t = 22t.$

$\int_a^b \vec{F}\cdot r'(t)\,dt = \int_0^4 22t\,dt = 11t^2 \Big|_0^4 = 11 \cdot 16 = \boxed{176.}$

$$\begin{array}{r} 16 \\ 16 \\ \hline 176 \end{array}$$

ex. $\varphi(x,y,z) = xy + xz + yz$. ← potential function. $\vec{F} = \nabla\varphi$.

C: $r(t) = \langle \underline{t, 2t, 3t} \rangle$. $0 \le t \le 4$.

b. If $F = \nabla\varphi$, then $\int_C F\cdot dr = \varphi(B) - \varphi(A)$.

$\because F = \nabla\varphi$. conservative $\therefore$ FTLT applies

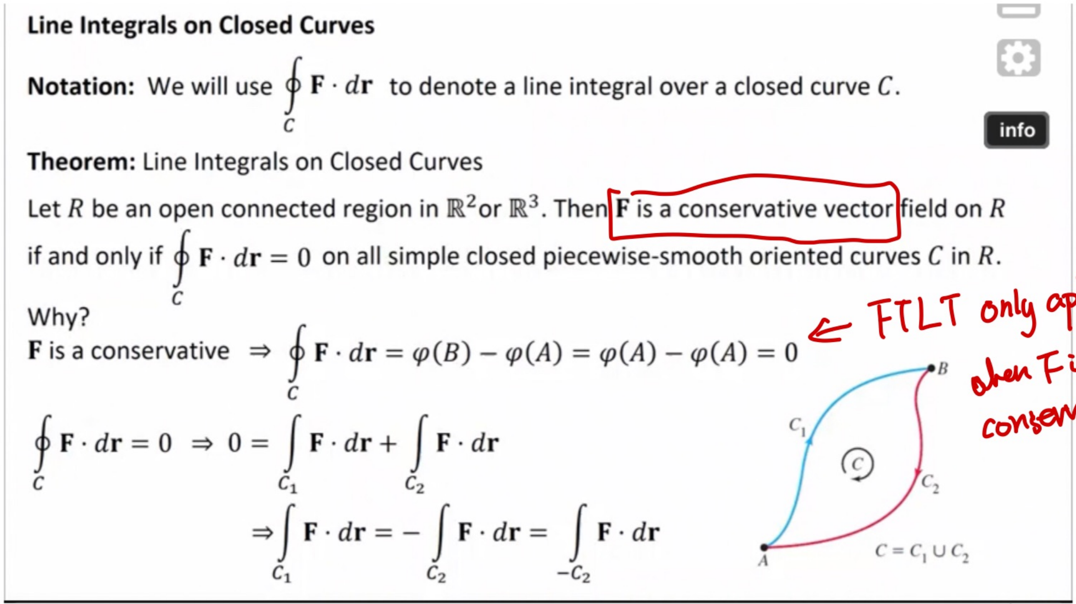
Also. A: $t=0$: $\vec{r}(0) = \langle 0, 0, 0 \rangle \Rightarrow A = \langle 0, 0, 0 \rangle$.

B: $t=4$: $\vec{r}(4) = \langle 4, 8, 12 \rangle \Rightarrow B = \langle 4, 8, 12)$

$\int_C \vec{F}\cdot d\vec{r} = \varphi(B) - \varphi(A) = \varphi(4,8,12) - \varphi(0,0,0)$

$= 4\cdot8 + 4\cdot12 + 8\cdot12 - 0\cdot0 - 0\cdot0 - 0\cdot0.$

$= \underline{32} + \underline{48} + 96 = 80 + 96 = \boxed{176}$



Line Integrals on Closed Curves

Notation: We will use $\oint_C \mathbf{F} \cdot d\mathbf{r}$ to denote a line integral over a closed curve C.

Theorem: Line Integrals on Closed Curves

Let R be an open connected region in $\mathbb{R}^2$or $\mathbb{R}^3$. Then $\mathbf{F}$ is a conservative vector field on R if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple closed piecewise-smooth oriented curves C in R.

Why?

$\mathbf{F}$ is a conservative $\Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A) = \varphi(A) - \varphi(A) = 0$ ← FTLT only apply when F is conservative.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \Rightarrow 0 = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

$$\Rightarrow \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = -\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{-C_2} \mathbf{F} \cdot d\mathbf{r}$$

$\vec{F} = \langle \overset{f}{e^x \cos y}, \overset{g}{-e^x \sin y} \rangle$.. $\int_C \vec{F} \cdot dr$: C: is triangular path from $\underset{A}{(0.0)} \rightarrow (-3.5)$ to $(-3:3)$ to $\underset{A}{(0.0)}$.

$f_y = -e^x \sin y = g_x$. $\Rightarrow$ F is conservative.

50 $\int_C \vec{F} \cdot d\vec{r} = \varphi(A) - \varphi(A) = 0$.

$= \varphi(0.0) - \varphi(0.0) = 0$

$\int_C \vec{F}\cdot dr$, $\vec{F} = \langle e^x \cos y, -e^x \sin y \rangle$, C is a triangular path from (0.0) to (-3,5) to (-3,-3) to (2.0).

$\vec{F}$ is conservative so

$\int_C \vec{F}\cdot dr = \varphi(B) - \varphi(A)$ $\qquad$ $\varphi = 2yz - xz^2 + x^2 y + c.$

Last pt (0.0) and first pt (0.0)

$= \varphi(0.0) - \varphi(0,0)$

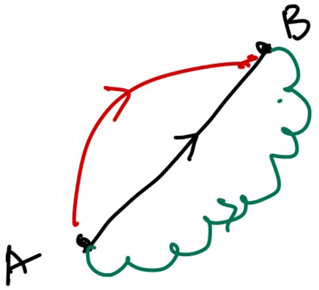
$= 0$

* Remember that FTL only works when F is conservative!

Theorem 17.5

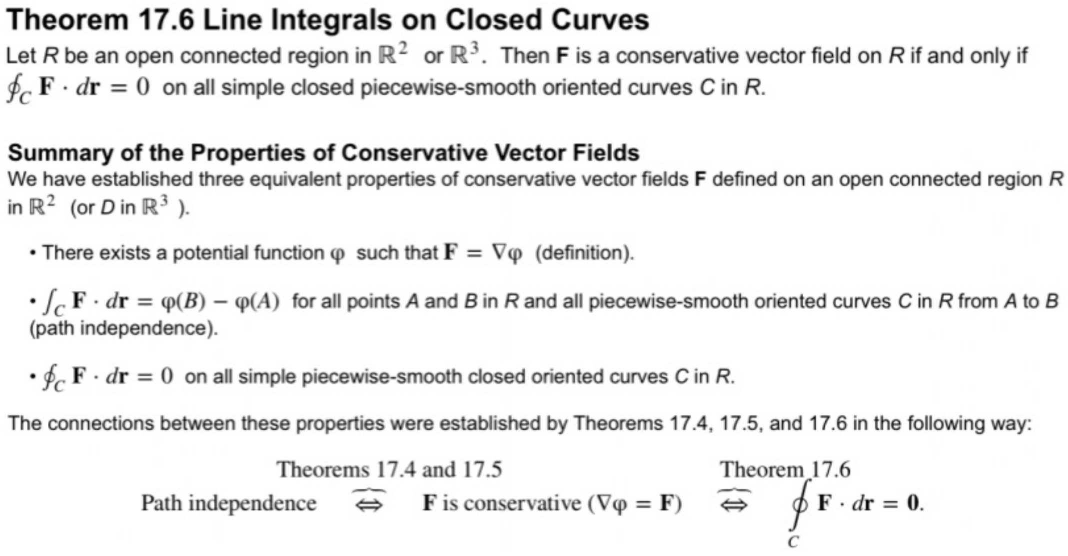
Let **F** be a continuous vector field on an open connected region *R* in $\mathbb{R}^2$. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, then **F** is conservative; that is, there exists a potential function φ such that $\mathbf{F} = \nabla\varphi$ on *R*.

F is conservative $\Leftrightarrow$ independent of path.



$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$.

C_1 and C_2.
has the same
initial and terminal pts.



Theorem 17.6 Line Integrals on Closed Curves

Let *R* be an open connected region in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then **F** is a conservative vector field on *R* if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple closed piecewise-smooth oriented curves *C* in *R*.

Summary of the Properties of Conservative Vector Fields

We have established three equivalent properties of conservative vector fields **F** defined on an open connected region *R* in $\mathbb{R}^2$ (or *D* in $\mathbb{R}^3$).

- There exists a potential function φ such that $\mathbf{F} = \nabla\varphi$ (definition).
- $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$ for all points *A* and *B* in *R* and all piecewise-smooth oriented curves *C* in *R* from *A* to *B* (path independence).
- $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ on all simple piecewise-smooth closed oriented curves *C* in *R*.

The connections between these properties were established by Theorems 17.4, 17.5, and 17.6 in the following way:

$$\text{Path independence} \quad \overset{\text{Theorems 17.4 and 17.5}}{\overset{\frown}{\Leftrightarrow}} \quad \mathbf{F} \text{ is conservative } (\nabla\varphi = \mathbf{F}) \quad \overset{\text{Theorem 17.6}}{\overset{\frown}{\Leftrightarrow}} \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = \mathbf{0}.$$

When $\vec{F}$ is conservative and C is closed,

$\int_C \vec{F} \cdot dr = 0$

"conservative" vector field is a property that will show up on Ch17 a lot

make sure you get use to it :)

17.4 Green's Theorem.

Definition Two-Dimensional Curl

The **two-dimensional curl** of the vector field $\mathbf{F} = \langle f, g \rangle$ is $\boxed{\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}}$ → 2D-curl. If the curl is zero throughout a region, the vector field is **irrotational** on that region.

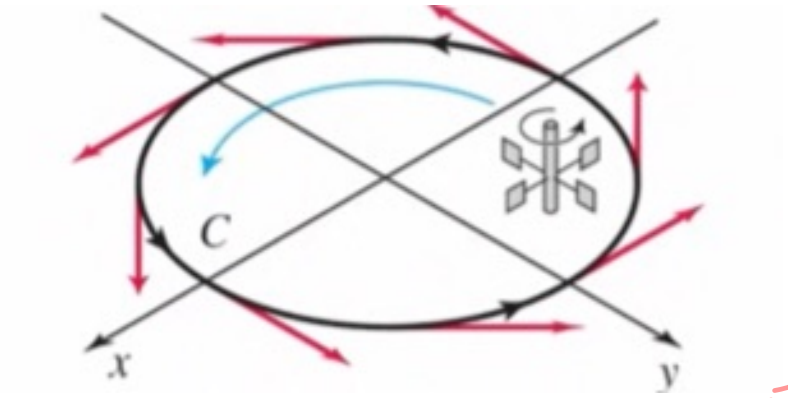
Theorem 17.7 Green's Theorem—Circulation Form

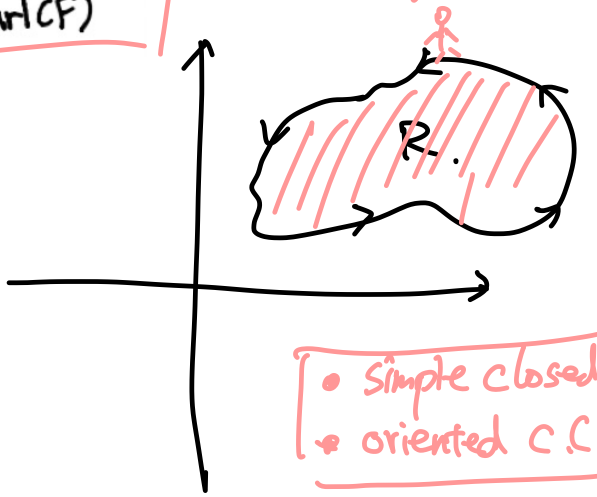
Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

the circulation on the boundary of R : C. ←

$$\underbrace{\boxed{\oint_C \mathbf{F} \cdot d\mathbf{r}}}_{\text{circulation}} = \underbrace{\oint_C f\,dx + g\,dy}_{\text{circulation}} = \boxed{\underset{\text{curl(F)}}{\underline{\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)}}\, dA.}$$

→ net rotations curl throughout the region R.



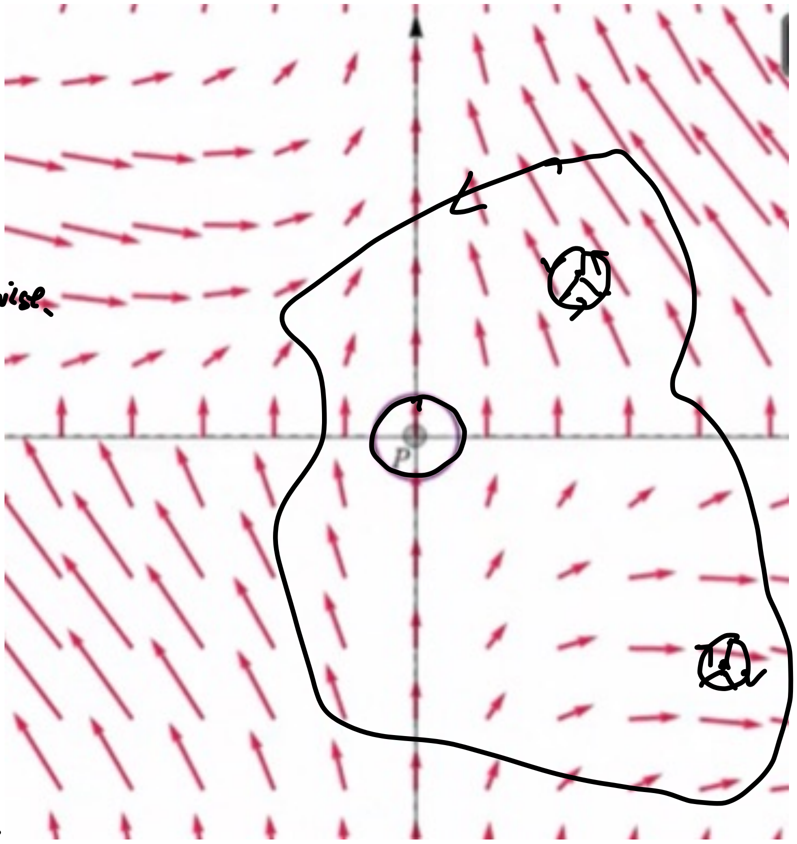


- simple closed C
- oriented C.C

$$\oint_C F \cdot dr = \iint_D \underline{g_x - f_y}\, dA.$$

2D-Curl.

- $\underline{g_x - f_y > 0}$ ⇒ counter clockwise.
- $g_x - f_y < 0$ ⇒ clockwise.
- $g_x - f_y = 0$ ⇒ no rotation.



- $\vec{F} = \langle f, g \rangle$ is conservative

⇒ $f_y = g_x$ ⇒ 2D. curl $= g_x - f_y = 0.$

⇒ $\boxed{\oint_C F \cdot dr = 0.}$

- If needed: $\underset{\boxed{-C}}{\oint} F \cdot dr = -\boxed{\int_C \vec{F} \cdot d\vec{v}}$

c ⇒ cc. $\boxed{-C}$ → clockwise.

e.x). $F = \langle \underbrace{-2xy}_{f}, \underbrace{x^2}_{g} \rangle$. R: bounded by $C: \begin{cases} y = x(2-x) \leftarrow \text{parabola} \\ y = 0. \end{cases}$

$$\boxed{\bullet \oint_C \vec{F}\cdot d\hat{r}} = \iint_R g_x - f_y \, dA = \boxed{\frac{16}{3}}$$

Green's Thm :

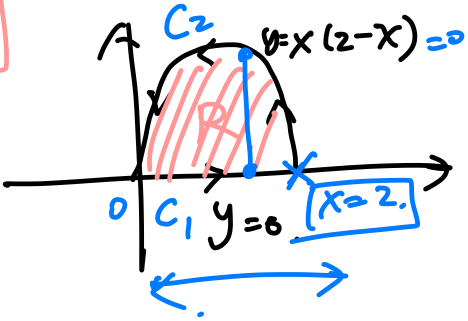
$$= \iint_R \frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}(-2xy)\, dA$$

$$= \iint_{R.} 2x - (-2x)\, dA = \iint_R 4x\, dA.$$

$$= \int_0^2 \int_0^{x(2-x)} 4x \quad dy\, \underline{dx}.$$

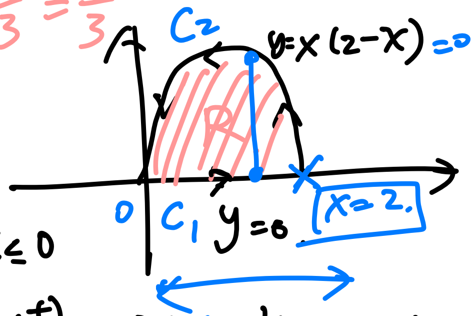
$$= \int_0^2 4xy \Big|_0^{x(2-x)} dx = \int_0^2 4x(x(2-x))\, dx$$

$$= \int_0^2 8x^2 - 4x^3\, dx = \frac{8}{3}x^3 - \frac{4}{4}x^4 \Big|_0^2 = \frac{8\cdot 8}{3} - 16 = \frac{64-48}{3} = \frac{16}{3}$$



e.x). $F = \langle \underbrace{-2xy}_{f}, \underbrace{x^2}_{g} \rangle.$ R: bounded by $C: \begin{cases} y = x(2-x) \leftarrow \text{parabola} \\ y = 0. \end{cases}$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot dr + \int_{C_2} \vec{F} \cdot dr. = 0 + \frac{16}{3} = \frac{16}{3}$$



C_1: line segment.

$r(t) = \langle t, 0 \rangle \quad 0 \le t \le 2.$

$r'(t) = \langle 1. 0 \rangle.$

$\vec{F}(\vec{r}(t)) = \langle -2xy. x^2 \rangle = \langle 0, t^2 \rangle$

$\vec{F} \cdot r'(t) = \langle 0. t^2 \rangle \cdot \langle 1. 0 \rangle = 0.$

$\int_{C_1} F \cdot r'(t) = \int_0^2 0 \, dt = 0.$

C_2: parabola

$x = -t. \quad -2 \le t \le 0$

$y = x(2-x) = -t(2+t) = -2t - t^2. \quad -2 \le t \le 0$

$r(t) = \langle -t, -2t - t^2 \rangle$

$r'(t) = \langle -1, -2 - 2t \rangle.$

$\vec{F}(\vec{r}(t)) = \langle -2xy. x^2 \rangle = \langle -2(-t)(-2t-t^2), t^2 \rangle$

$= \langle -4t^2 - 2t^3, t^2 \rangle$

$\vec{F} \cdot r'(t) = \langle -4t^2 - 2t^3, t^2 \rangle \cdot \langle -1, -2-2t \rangle.$

$= 4t^2 \cancel{+2t^3} - 2t^2 \cancel{-2t^3}.$

$= 2t^2. \quad \int_{C_2} F \cdot r' dt = \int_{-2}^{0} 2t^2 dt = \frac{2}{3} t^3 \Big|_{-2}^{0} = \frac{16}{3}.$

e.x) $\vec{F} = \langle \underset{f}{\underline{x+y}}, \underset{g}{\underline{2x^2}} \rangle$ C: is triangular. from

(0.0) → (5,0) → (5.5) → (0.0).

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R g_x - f_y \, dA$$

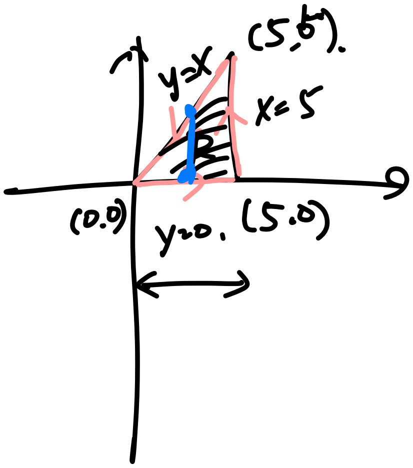
Green's Thm $= \iint_{R.} 4x - 1 \ dA$

$$= \int_0^5 \int_0^x 4x - 1. \ dy\,dx$$

$$= \int_0^5 4xy - y \Big|_0^x \ dx$$

$$= \int_0^5 4x^2 - x \ dx$$

$$= \frac{4}{3}x^3 - \frac{1}{2}x^2 \Big|_0^5 = \boxed{\frac{4\cdot5^3}{3} - \frac{25}{2}}.$$



Theorem 17.8 Area of a Plane Region by Line Integrals

Under the conditions of Green's Theorem, the area of a region R enclosed by a curve C is

$$\text{①}\ \boxed{\oint_C x\,dy} = \overset{\text{②}}{\boxed{-\oint_C y\,dx}} = \overset{\text{③}}{\boxed{\frac{1}{2}\oint_C (x\,dy - y\,dx)}}.\ \bigstar$$

$$\oint_C F\cdot dr = \iint_R \underline{g_x - f_y}\, dA = \iint_R 1\, dA = \boxed{\text{Area of Region } R}$$

2D-curl $\boxed{g_x - f_y = 1.}$

$F: \langle f, g\rangle = \langle \underline{0, x}\rangle \Rightarrow g_x - f_y = 1 - 0 = 1. \Rightarrow \oint_C F\,dr = \oint_C \cancel{f\,dx} + g\,dy$

$F: \langle f, g\rangle = \langle \underline{-y, 0}\rangle \Rightarrow g_x - f_y = 0 - (-1) = 1 \Rightarrow \oint_C F\,dr = \oint_C f\,dx + \cancel{g\,dy} = \boxed{\oint_C x\,dy}$

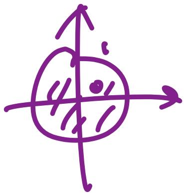
$= -\oint_C y\,dx$

$= \iint_R 1\,dA = \text{Area}.$

$F: \langle f, g\rangle = \langle \underline{-\tfrac{1}{2}y, \tfrac{1}{2}x}\rangle \Rightarrow g_x - f_y = \tfrac{1}{2} - (-\tfrac{1}{2}) = 1 \Rightarrow \oint_C F\,dr = \oint f\,dx + g\,dy = \oint_C -\tfrac{1}{2}y\,dx + \tfrac{1}{2}x\,dy$

$= \tfrac{1}{2}\oint_C x\,dy - y\,dx.$

e.x).



$r(t) = \langle \cos t, \sin t \rangle.$ $\boxed{\pi.}$

$x = \cos t.$ $dx = -\sin t\,dt.$

$y = \sin t$ $dy = \cos t\,dt.$

$0 \le t \le 2\pi$

$$\text{Area of } R = \frac{1}{2} \oint_C (x\,dy - y\,dx) = \oint_c x\,dy = -\oint_c y\,dx$$

$$= \frac{1}{2} \int_0^{2\pi} \cos t \cdot \cos t\,dt - \sin t \cdot (-\sin t)\,dt.$$

$$= \frac{1}{2} \int_0^{2\pi} \cos^2 t + \sin^2 t \; dt$$

$$= \frac{1}{2} \int_0^{2\pi} 1\,dt$$

$$= \frac{1}{2} 2\pi = \boxed{\pi}.$$

$\boxed{\cos^2 t + \sin^2 t = 1}$

C: Ellipse : $\frac{x^2}{36} + \frac{y^2}{16} = 1.$

$r(t) = \langle 6\cos t, 4\sin t \rangle$

$x = 6\cos t$ $y = 4\sin t$ $0 \le t \le 2\pi$

$\boxed{ab\pi \\ = 4 \cdot 6\pi \\ = 24\pi}$

$\text{Area of } R = \frac{1}{2} \oint_c x\,dy - y\,dx$

$dx = -6\sin t\,dt$

$dy = 4\cos t\,dt.$

$$= \frac{1}{2} \int_0^{2\pi} 6\cos t\, 4\cos t\,dt - 4\sin t(-6\sin t)\,dt$$

$$= \frac{1}{2} \int_0^{2\pi} 24\left[\cos^2 t + \sin^2 t\right] dt$$

$$= \frac{1}{2} \int_0^{2\pi} 24\,dt = \frac{1}{2} \cdot 24 \cdot 2\pi = \boxed{24\pi}$$

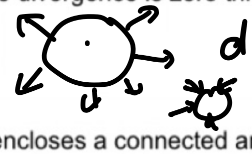
continue 17.4. Green's Thm.

$\vec{F}=\langle f, g\rangle$.

2D-divergence $\Rightarrow f_x+g_y$.

Definition Two-Dimensional Divergence

The **two-dimensional divergence** of the vector field $\mathbf{F} = \langle f, g\rangle$ is $\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}$. If the divergence is zero throughout a region, the vector field is **source free** on that region.



$\text{div}=f_x+g_y>0$

$\text{div}<0$

Theorem 17.9 Green's Theorem—Flux Form

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $\mathbf{F} = \langle f, g\rangle$, where f and g have continuous first partial derivatives in R. Then

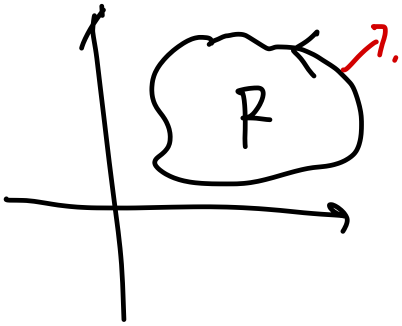
$\vec{n}=\langle y', -x'\rangle$

flux across the boundary of R.

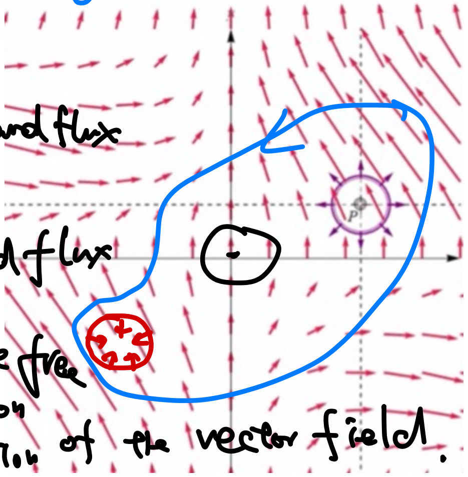
$$\underbrace{\boxed{\oint_C \mathbf{F}\cdot\mathbf{n}\,ds}}_{\text{outward flux}} = \underbrace{\oint_C f\,dy - g\,dx}_{\text{outward flux}} = \boxed{\iint_R \underbrace{\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}\right)}_{\text{div}(F)\ \|\ 0} dA,}$$

net divergence throughout (*) the region R.

where **n** is the outward unit normal vector on the curve.

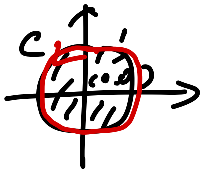


- $f_x+g_y>0$ outward flux
- $f_x+g_y<0$ inward flux
- $\boxed{f_x+g_y=0} \Rightarrow$ source free



divergence measure < expansion / contraction of the vector field.

ex) $F = \langle x^2+y^2, 0\rangle.$ $R = \{(x,y) : x^2+y^2 \le 1\}$



$\boxed{\oint_C \vec{F}\cdot\vec{n}\,ds = 0.} = \boxed{\iint_R f_x + g_y \, dA.}$

$\vec{n} = \langle y', -x'\rangle. = \langle \cos t, \sin t\rangle.$

$\vec{r}(t) = \langle \underbrace{\cos t}_{x}, \underbrace{\sin t}_{y.}\rangle.$ $0 \le t \le 2\pi$

$\vec{r}'(t) = \langle -\sin t, \cos t\rangle.$ $x' = -\sin t$, $y' = \cos t.$

$\vec{F}(\vec{r}(t)) = \langle \cos^2 t + \sin^2 t, 0\rangle = \langle \underset{f}{1}. \underset{g.}{0}\rangle$

$\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C f\,dy - g\,dx = \int_0^{2\pi} 1\cdot\cos t\,dt. \cancel{-0(-\sin t)dt}$

$= \int_0^{2\pi} \cos t\,dt = \sin t \Big|_0^{2\pi} = 0 - 0 = 0.$

ex) $F = \langle \underset{f}{\underline{x^2+y^2}}, \underset{g}{\underline{0}} \rangle.$

$R = \{(x,y) : x^2+y^2 \le 1\}$

$$\iint_R f_x + g_y \, dA$$

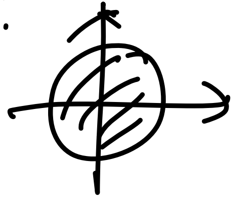
$dA = r\,dr\,d\theta.$

$R = \{(r,\theta) : 0 \le r \le 1, 0 \le \theta \le 2\pi\}$

$$= \int_0^{2\pi}\int_0^1 2x(r\,dr\,d\theta).$$

$f_x + g_y = 2x + 0 = 2x.$

$x = r\cos\theta.$



$$= \int_0^{2\pi}\int_0^1 2r\cos\theta \, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \cos\theta \left[\int_0^1 2r^2\,dr\right] d\theta$$

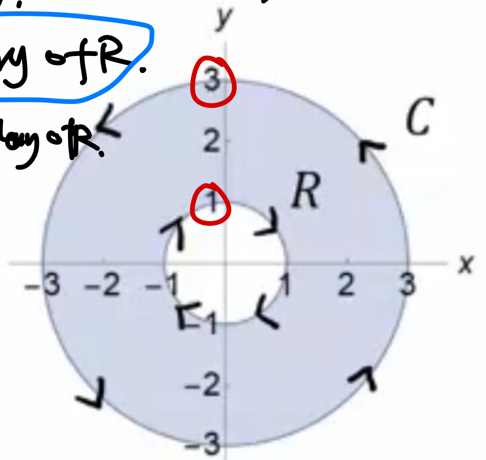
$$= \int_0^{2\pi} \cos\theta \left[\tfrac{2}{3}r^3 \Big|_0^1\right] d\theta$$

$$= \int_0^{2\pi} \cos\theta \cdot \frac{2}{3}\,d\theta = \frac{2}{3}\sin\theta \Big|_0^{2\pi} = 0 - 0 = 0.$$

ex). $F = \langle \underset{f}{-y}, \underset{g}{x} \rangle$. $R: \{(r,\theta),\ 1 \le r \le 3,\ 0 \le \theta \le 2\pi\}$.

a. compute the circulation on the boundary of R.

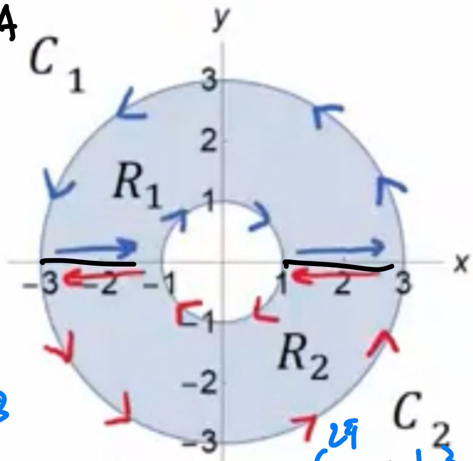
b. compute the flux across the boundary of R.



$$\oint_C F\cdot dr = \underbrace{\oint_{C_1} F\cdot dr}_{\downarrow \text{ Green's}} + \underbrace{\int_{C_2} F\cdot dr}_{\downarrow \text{ Green's}}.$$

$$= \iint_{R_1} g_x - f_y\, dA + \iint_{R_2} g_x - f_y\, dA$$

$$= \iint_R \boxed{g_x - f_y}\, dA.$$



$f_y = \frac{\partial}{\partial y}(-y) = -1$ $\quad g_x = \frac{\partial}{\partial x} x = 1.$

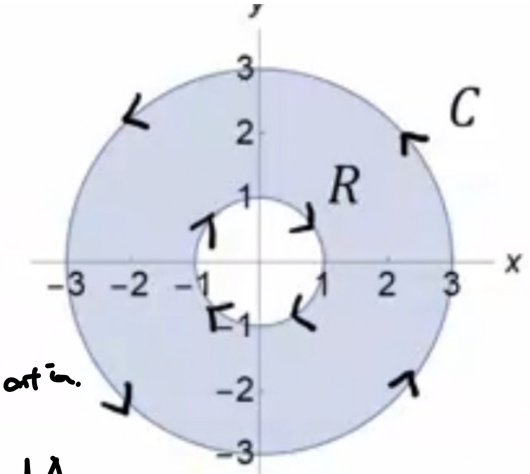
$g_x - f_y = 1-(-1) = 2.$

$$\iint 2\, dA = \int_0^{2\pi}\int_1^3 2\, r\,dr\,d\theta = \int_0^{2\pi} r^2\Big|_1^3 d\theta = 2\pi(9-1) = 2\pi\cdot 8 = 16\pi.$$

$$F = \langle \underbrace{-y}_{f}, \underbrace{x}_{g.} \rangle$$

Green's Thm

$$\oint_c F \cdot n \, ds = \iint_R \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \, dA$$



↑ By similar argument to the circulation.

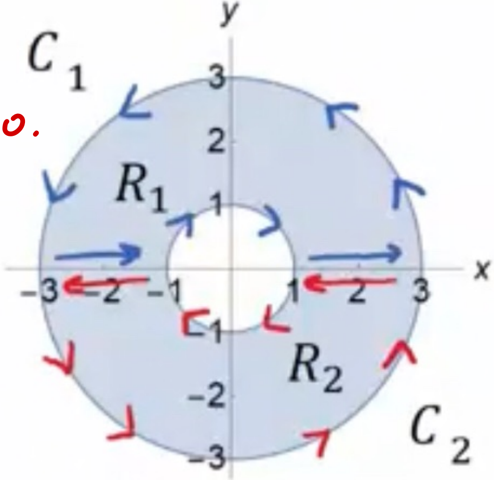
$$\oint_c \vec{F} \cdot \vec{n} \, ds = \iint_R f_x + g_y \, dA = \iint \underbrace{0 + 0}_{\downarrow} \, dA$$

div(F) = 0.

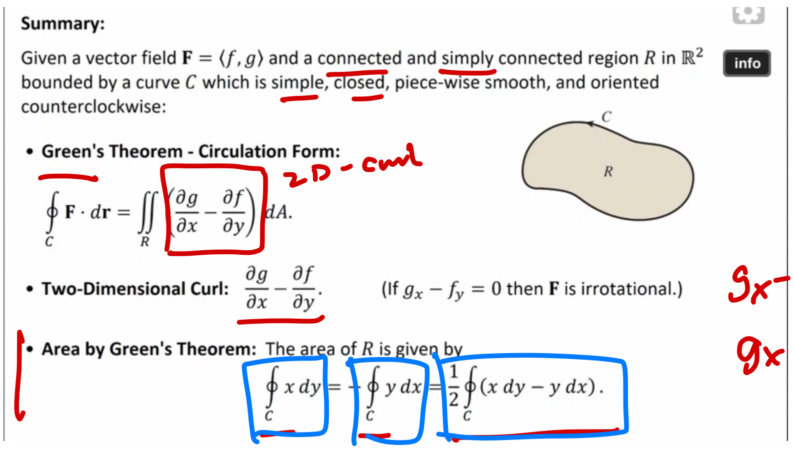
$$= \int_0^{2\pi} \int_1^3 0 \, (r \, dr \, d\theta)$$

$$= 0$$

⟸ net flux is zero across C.



17.4.



Summary:

Given a vector field $\mathbf{F} = \langle f, g \rangle$ and a connected and simply connected region R in $\mathbb{R}^2$ bounded by a curve C which is simple, closed, piece-wise smooth, and oriented counterclockwise:

- **Green's Theorem - Circulation Form:**

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \boxed{\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)} dA.$$

2D - curl

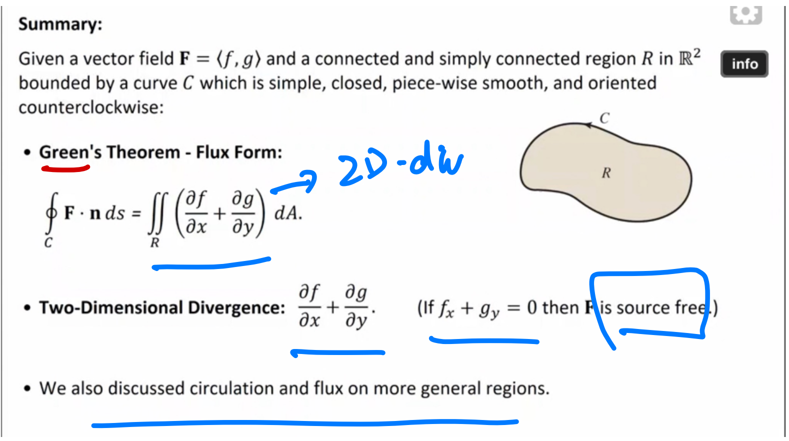
- **Two-Dimensional Curl:** $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$. (If $g_x - f_y = 0$ then $\mathbf{F}$ is irrotational.)

- **Area by Green's Theorem:** The area of R is given by

$$\boxed{\oint_C x\,dy} = -\boxed{\oint_C y\,dx} = \boxed{\frac{1}{2}\oint_C (x\,dy - y\,dx)}.$$

$g_x - f_y > 0$.

$g_x - f_y < 0$



Summary:

Given a vector field $\mathbf{F} = \langle f, g \rangle$ and a connected and simply connected region R in $\mathbb{R}^2$ bounded by a curve C which is simple, closed, piece-wise smooth, and oriented counterclockwise:

- **Green's Theorem - Flux Form:**

$$\oint_C \mathbf{F} \cdot \mathbf{n}\,ds = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA.$$

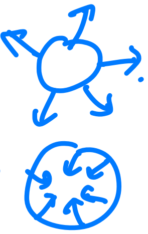
→ 2D - div

- **Two-Dimensional Divergence:** $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$. (If $f_x + g_y = 0$ then $\mathbf{F}$ is source free.)

- We also discussed circulation and flux on more general regions.

$f_x + g_y > 0$

$f_x + g_y < 0$



17.5. Divergence and Curl.

Recall 2D-div. of F. $\text{div}(F) = f_x + g_y.$ ⇐ scalar valued function.

$F = \langle f, g \rangle.$

Definition Divergence of a Vector Field

The **divergence** of a vector field $\mathbf{F} = \langle f, g, \boxed{h} \rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\boxed{\text{div}\,\mathbf{F}} = \nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \boxed{\frac{\partial h}{\partial z}}.$$

If $\nabla \cdot \mathbf{F} = 0,$ the vector field is **source free**.

Del operator.

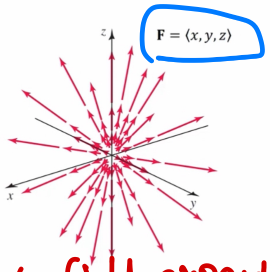
$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle.$$

$$\nabla \cdot F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f, g, h \rangle = \text{div}(F).$$

dot product.

$$= \frac{\partial}{\partial x} f + \frac{\partial}{\partial y} g + \frac{\partial}{\partial z} h$$

$$= f_x + g_y + h_z.$$



$$F = \langle x, y, z \rangle. \quad \text{div}(F) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1 = \boxed{3 > 0}$$

positive

vector field expands outward everywhere

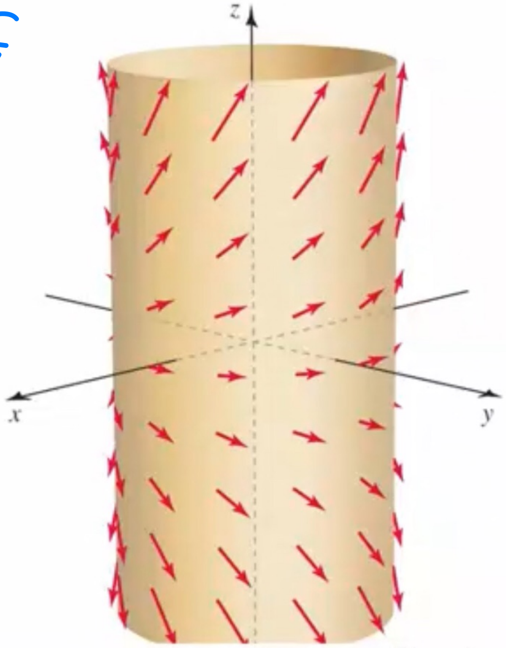
$F = \langle -y, x-z, y \rangle.$

$$\text{div}(F) = \frac{\partial -y}{\partial x} + \frac{\partial x-z}{\partial y} + \frac{\partial y}{\partial z}$$

$$= 0 + 0 + 0$$

$$= 0$$

Source free and neither expanding or contracting.



$\mathbf{F} = \langle -y, x - z, y \rangle$

17.5 continon

Definition Divergence of a Vector Field

The **divergence** of a vector field $\mathbf{F} = \langle f, g, h \rangle$ that is differentiable on a region of $\mathbb{R}^3$ is

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

If $\nabla \cdot \mathbf{F} = 0$, the vector field is **source free**.

ex). $F = \langle 2xy + z^2, x^2, 4xz^2 \rangle$.

$\langle f, \overset{\downarrow}{g}, h \rangle$

- Find $\text{div}(\vec{F})$
- state whether the vector field F is source free.

$$\text{div}(F) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$f = 2xy + z^2 \Rightarrow f_x = \frac{\partial\, 2xy + z^2}{\partial x} = 2y.$$

$$g = x^2 \Rightarrow g_y = \frac{\partial x^2}{\partial y} = 0.$$

$$h = 4xz^2 \Rightarrow h_z = \frac{\partial\, 4xz^2}{\partial z} = 8xz.$$

$$\text{div}(\vec{F}) = 2y + 0 + 8xz \neq 0.$$

$\Rightarrow$ F is <u>not</u> source free.

ex) Find the divergence of the radial vector field.

$$F = \frac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^2} = \frac{r}{|r|^{④}} = \langle f, g, h\rangle \qquad r = \langle x, y, z\rangle$$

$$|r| = \sqrt{x^2+y^2+z^2}$$ → length of r

$$\boxed{\operatorname{div}(\vec{F}) = \frac{-1}{|r|^4}} \checkmark$$

$$\operatorname{div}(F) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$f = \frac{x}{(x^2+y^2+z^2)^2} \Rightarrow f_x = \frac{1\cdot(x^2+y^2+z^2)^{\cancel{2}} - x\,2\cancel{(x^2+y^2+z^2)}\cdot 2x}{(x^2+y^2+z^2)^{\cancel{4}\,3}}$$

$$= \frac{x^2+y^2+z^2-4x^2}{(x^2+y^2+z^2)^3} = \frac{y^2+z^2-3x^2}{(x^2+y^2+z^2)^3}$$

$$g = \frac{y}{(x^2+y^2+z^2)^2} \Rightarrow g_y = \frac{x^2+z^2-3y^2}{(x^2+y^2+z^2)^3}$$

$$h = \frac{z}{(x^2+y^2+z^2)^2} \Rightarrow h_z = \frac{x^2+y^2-3z^2}{(x^2+y^2+z^2)^3}$$

$$\operatorname{div}(F) = f_x + g_y + h_z = \frac{y^2+z^2-3x^2+x^2+z^2-3y^2+x^2+y^2-3z^2}{(x^2+y^2+z^2)^3} = \frac{2(x^2+y^2+z^2)-3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\cancel{3}\,2}} = \frac{-1}{(x^2+y^2+z^2)^2} = \frac{-1}{|r|^4}$$

$$F = \frac{\langle x, y, z \rangle}{(x^2+y^2+z^2)^{p/2}} = \frac{r.}{|r|^p}$$

$$\operatorname{div}(F) = \frac{3-p}{|r|^p.} = \nabla \cdot F$$

2D - curl $F = \langle f . g \rangle$

$$\boxed{g_x - f_y}$$

Definition Curl of a Vector Field

The **curl** of a vector field $\underline{\mathbf{F} = \langle f, g, h \rangle}$ that is differentiable on a region of $\mathbb{R}^3$ is

measure the rotation of the vector field on each pts. ← $\nabla \times \mathbf{F} = \underline{\text{curl } \mathbf{F}}$

$$= \underbrace{\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}.}_{} \; *$$

If $\nabla \times \mathbf{F} = \mathbf{0}$, the vector field is **irrotational**.

$$\text{curl}(F) = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ g & h \end{vmatrix} \vec{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ f & h \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ f & g \end{vmatrix} \vec{k}$$

$$= (h_y - g_z)\vec{i} - (h_x - f_z)\vec{j} + \boxed{(g_x - f_y)}\vec{k}.$$

$$= \langle h_y - g_z,\ f_z - h_x,\ g_x - f_y \rangle.$$

ex) $F = \langle -y, \; x-z, \; +y \rangle$

$\langle \overline{f}, \; \overline{g}, \; \overline{h} \rangle$

$\text{curl} F = \nabla \times F = \nabla \times \langle -y, x-z, y \rangle.$

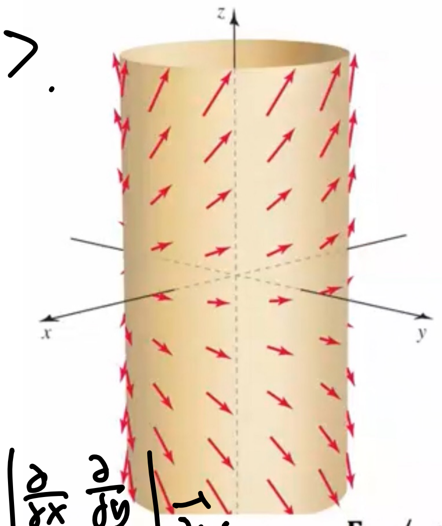
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x-z & y \end{vmatrix} =$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-z & y \end{vmatrix} \vec{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ -y & y \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ -y & x-z \end{vmatrix} \vec{k}.$$

$$= 1-(-1)\vec{i} - (0+0)\vec{j} + (1-(-1))\vec{k}$$

$$= \langle 2, 0, \boxed{2} \rangle \neq \vec{0} = (0.0.0)$$

↪ 2 in $\vec{k}$ gives the rotation of F in xy-plane.



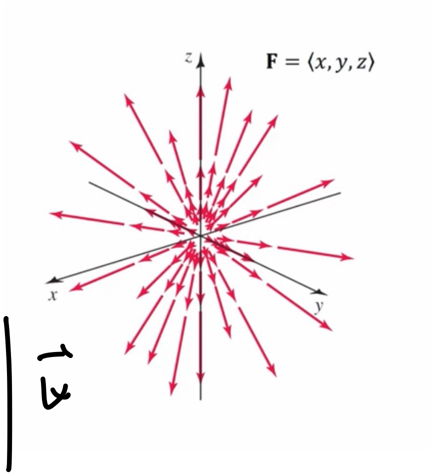
$\mathbf{F} = \langle -y, x-z, y \rangle$

ex) $F = \langle x . y . z)$ ← radial vector field. (curl $\underline{\vec{F} = \vec{0} = (0.0.0)}$).

$$\text{curl} F = \nabla \times F = \nabla \times \langle x . y , z \rangle$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} =$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z \end{vmatrix} i - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x & z \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x & y \end{vmatrix} \vec{k}$$



$$= (0-0) i - (0-0) \vec{j} + (0-0) \vec{k}$$

$$= \langle 0 . 0 . 0 \rangle$$

$$= \vec{0} . \quad \Rightarrow \hat{F} \text{ is irrotational.}$$

Thm : Curl of conservative vector field.

$$\text{Curl}(\vec{F}) = \vec{0}. \quad \text{if } \boxed{\vec{F} = \nabla\varphi.}$$

⟺ "if $\vec{F}$" is conservative ⟹ $\vec{F}$ is irrotational.

~~proof~~ proof : $F = \nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z \rangle$.

$$\text{curl} F = \nabla \times F = \boxed{\nabla \times \langle \nabla \cdot \varphi \rangle} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi_x & \varphi_y & \varphi_z \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi_y & \varphi_z \end{vmatrix} \vec{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \varphi_x & \varphi_z \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \varphi_x & \varphi_y \end{vmatrix} \vec{k}$$

φ has continuous 2nd partial derivative. →

$$= (\varphi_{zy} - \underline{\varphi_{yz}})\vec{i} - (\varphi_{zx} - \varphi_{xz})\vec{j} + (\varphi_{yx} - \varphi_{xy})\vec{k}$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k} = \langle 0, 0, 0 \rangle = \vec{0}.$$

What is $\nabla \cdot (\nabla \times F) = \text{div}(\text{curl} F)$. ?

(divergence of a curl of a vector field).

Thm of Divergence of curl. $\text{div}\,\underline{\text{curl}(F)} = 0$

$$\boxed{\text{curl}\,(\underline{\text{div} F}).}$$

$\swarrow$ not defined.

$\hookrightarrow$ scalar.

proof: $F = \langle f, g, h \rangle$.

The $\nabla \times F = \text{curl}\, F = \langle h_y - g_z, f_z - h_x, g_x - f_y \rangle$

Now $\nabla \cdot (\nabla \times F) = \text{div}(\text{curl}\, \vec{F})$.

$= \text{div}\langle h_y - g_z, f_z - h_x, g_x - f_y \rangle$

$= (\cancel{h_{yx}} - \underline{g_{zx}}) + (\underline{\underline{f_{zy}}} - \cancel{h_{xy}}) + (\underline{g_{xz}} - \underline{\underline{f_{yz}}})$.

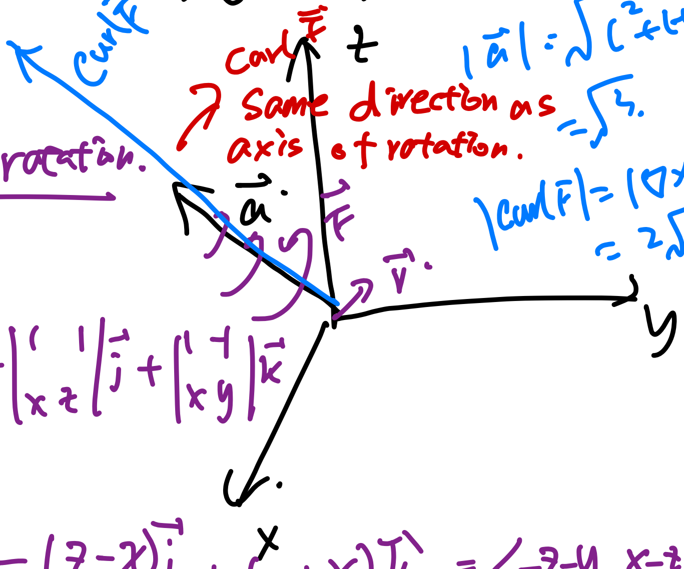
$= 0$ (constant. not a vector).

* assume f.g.h (
continous partial,
⇒ order of partial
does not matter

Ex) $\vec{F} = \underbrace{\langle 1, -1, 1\rangle}_{\vec{a}} \times \vec{r}$, $\vec{r} = \langle x, y, z\rangle$.

$\vec{F} = \vec{a} \times \vec{r} \Leftarrow$ a is axis of rotation.

is $\langle 1, -1, 1\rangle$



$|\vec{a}| = \sqrt{1^2 + (-1)^2 + 1^2}$
$= \sqrt{3}.$

$|\text{curl} \vec{F}| = |\nabla \times \vec{F}|$
$= 2\sqrt{3}.$

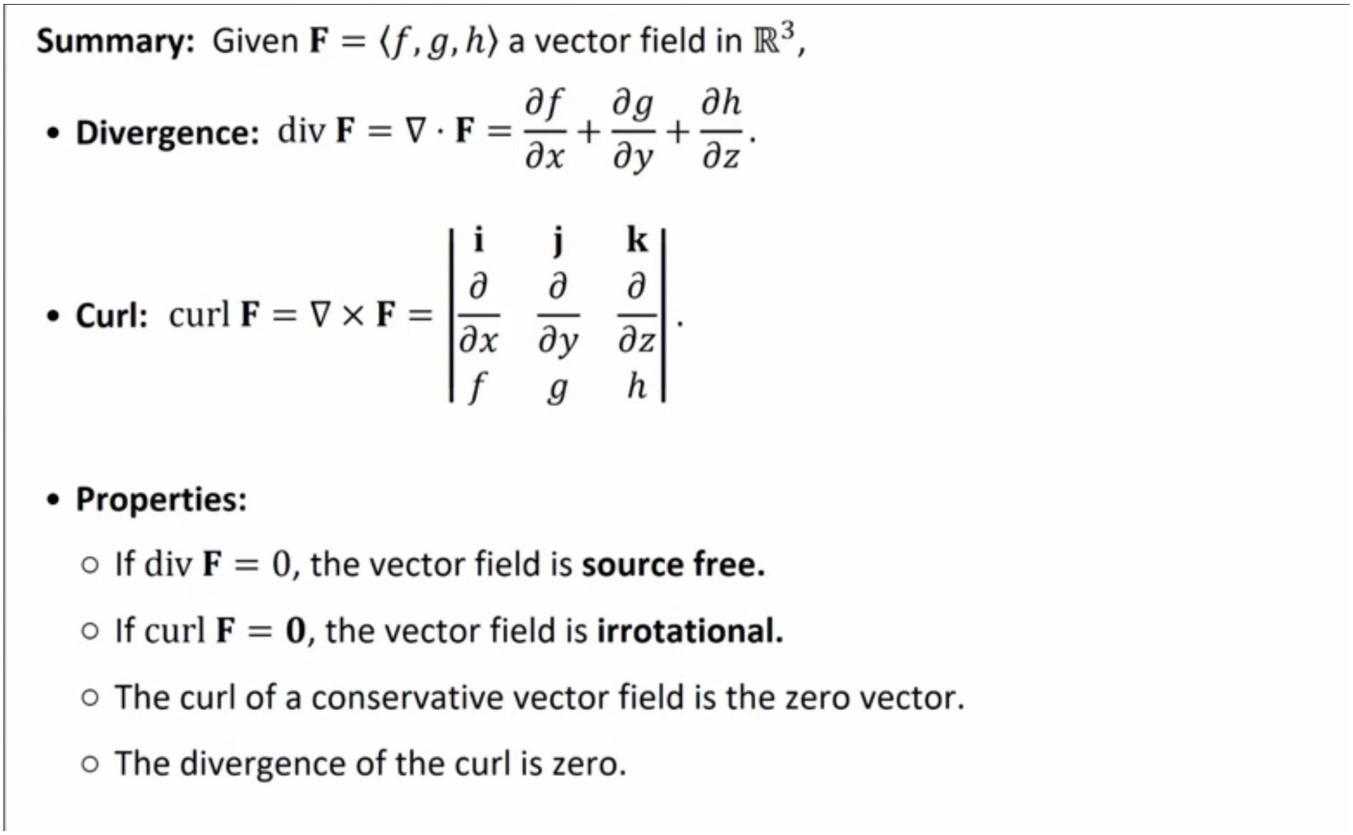
$$\vec{F} = \begin{vmatrix} \vec i & \vec j & \vec k \\ 1 & -1 & 1 \\ x & y & z \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ y & z \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 1 \\ x & z \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & -1 \\ x & y \end{vmatrix} \vec{k}$$

$$= (-z - y)\vec{i} - (z - x)\vec{j} + (y + x)\vec{k} = \langle -z-y, x-z, x+y \rangle.$$

$$\text{curl} \vec{F} = \begin{vmatrix} \vec i & \vec j & \vec k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z-y & x-z & x+y \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-z & x+y \end{vmatrix} \vec{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ -z-y & x+y \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ -z-y & x-z \end{vmatrix} \vec{k}$$

$$= (1-(-1))\vec{i} - (1-(-1))\vec{j} + (1-(-1))\vec{k} = (2, -2, 2) = 2\langle 1, -1, 1\rangle$$

$= 2\vec{a}$. curl $\vec{F} \parallel \vec{a} \rightarrow$ $|\text{curl} \vec{F}| = 2|\vec{a}|$.

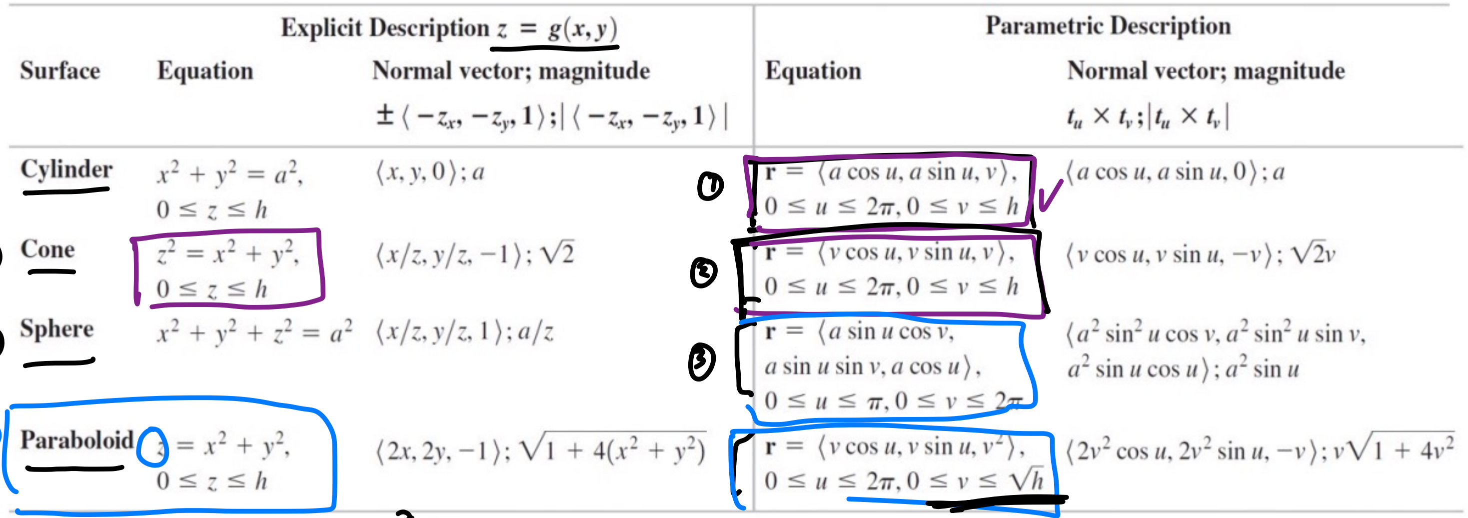


Summary: Given $\mathbf{F} = \langle f, g, h \rangle$ a vector field in $\mathbb{R}^3$,

- **Divergence:** $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} + \dfrac{\partial h}{\partial z}$.
- **Curl:** $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$.
- **Properties:**
  - If $\operatorname{div} \mathbf{F} = 0$, the vector field is **source free.**
  - If $\operatorname{curl} \mathbf{F} = \mathbf{0}$, the vector field is **irrotational.**
  - The curl of a conservative vector field is the zero vector.
  - The divergence of the curl is zero.

17.6. Surface integral.

- Parameterized Surface.



	Explicit Description $\underline{z = g(x, y)}$		Parametric Description	
Surface	Equation	Normal vector; magnitude $\pm\langle -z_x, -z_y, 1\rangle; \lvert\langle -z_x, -z_y, 1\rangle\rvert$	Equation	Normal vector; magnitude $t_u \times t_v; \lvert t_u \times t_v\rvert$
① Cylinder	$x^2 + y^2 = a^2$, $0 \le z \le h$	$\langle x, y, 0\rangle; a$	① $\mathbf{r} = \langle a\cos u, a\sin u, v\rangle$, $0 \le u \le 2\pi, 0 \le v \le h$ ✓	$\langle a\cos u, a\sin u, 0\rangle; a$
② Cone	$z^2 = x^2 + y^2$, $0 \le z \le h$	$\langle x/z, y/z, -1\rangle; \sqrt{2}$	② $\mathbf{r} = \langle v\cos u, v\sin u, v\rangle$, $0 \le u \le 2\pi, 0 \le v \le h$	$\langle v\cos u, v\sin u, -v\rangle; \sqrt{2}v$
③ Sphere	$x^2 + y^2 + z^2 = a^2$	$\langle x/z, y/z, 1\rangle; a/z$	③ $\mathbf{r} = \langle a\sin u\cos v$, $a\sin u\sin v, a\cos u\rangle$, $0 \le u \le \pi, 0 \le v \le 2\pi$	$\langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v$, $a^2\sin u\cos u\rangle; a^2\sin u$
④ Paraboloid	$z = x^2 + y^2$, $0 \le z \le h$	$\langle 2x, 2y, -1\rangle; \sqrt{1 + 4(x^2 + y^2)}$	$\mathbf{r} = \langle v\cos u, v\sin u, v^2\rangle$, $0 \le u \le 2\pi, 0 \le v \le \sqrt{h}$	$\langle 2v^2\cos u, 2v^2\sin u, -v\rangle; v\sqrt{1 + 4v^2}$

Recall: A curve $\mathbb{R}^2$, is defined parameterially $\vec{r}(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$.

For a surface in $\mathbb{R}^3$, we'll need two parameters and 3 dependent variables: $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$

$R = \{(u,v).\ a \le u \le b, c \le v \le d\}$

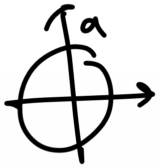
Parameterized Surface.

① Cylinder radius $a > 0$.

height $h > 0$.

$$\boxed{x^2+y^2=a^2. \quad 0\le z\le h}$$

(axis along z-axis).



$x = a\cos\theta$

$y = a\sin\theta$.

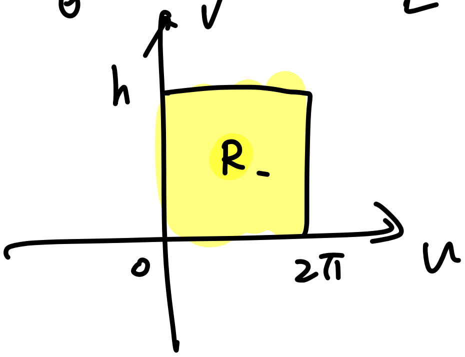
$z = z$.

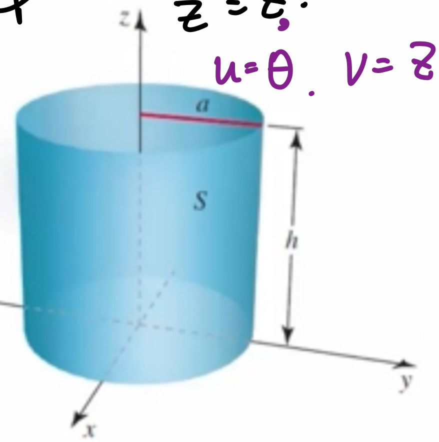
$u=\theta$. $v=z$.

Parametric Description:

$$\boxed{r(u.v). = \langle a\cos u,\ a\sin u,\ v\rangle}$$

$0 \le \underset{\theta}{\underset{\vee}{u}} \le 2\pi \qquad 0 \le \underset{z}{\underset{\parallel}{v}} \le h$





$x = a\cos u$

$y = a\sin u$

$z = v$.

$R = \{(u.v) : 0 \le u \le 2\pi,\ 0 \le v \le h\}$

② **cone.** • radius $a>0$. (axis is z-axis).

$$\boxed{\begin{array}{l} z^2 = x^2+y^2. \\ 0 \le z \le h \end{array}}$$

$z=h$.
$x^2+y^2=h^2$.

$\Rightarrow a=h$.

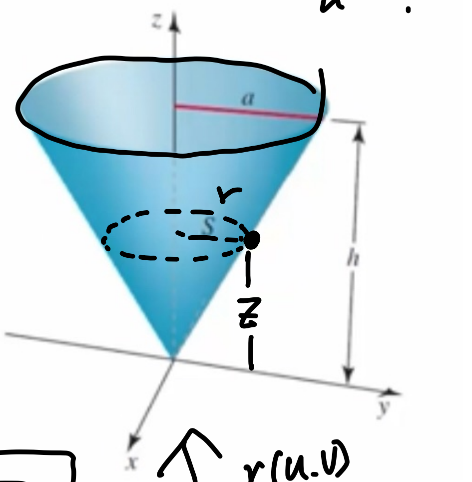
$= \frac{a}{h} = 1$.

• height $h>0$.

• (vertex at origin).

$$\boxed{\frac{r}{z} = \frac{a}{h}} \Rightarrow r = \frac{a}{h}\cdot z \quad \text{or} \quad z = \frac{hr}{a}.$$

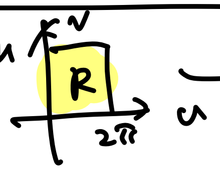
$$\begin{cases} x = r\cos\theta = \frac{az}{h}\cos\boxed{\theta} \\ y = r\sin\theta. = \frac{az}{h}\sin\boxed{\theta}. \\ z = z. \end{cases}$$



$u = \theta$. $\quad v = z$.

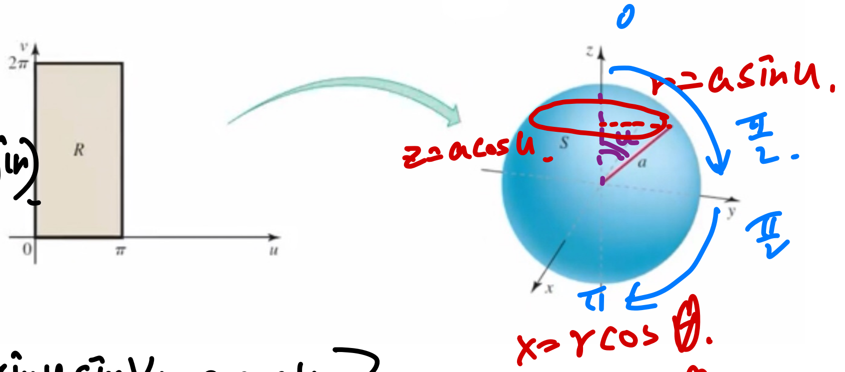
$$\boxed{r(u.v) = \left\langle \frac{av}{h}\cos u,\ \frac{av}{h}\sin u,\ v. \right\rangle}$$

$0 \le \underset{\theta}{\overset{}{u}} \le 2\pi \qquad 0 \le v. \le h.$



$r(u.v)$
$= \langle v\cos u, v\sin u, v \rangle$.

③ sphere. radius $a>0$.
(centered at origin)



$x = r\cos\theta.$

$y = r\sin\theta.$

parametric description.

$$r(u.v) = \langle \underbrace{a\sin u}_{r} \cos v, \underbrace{a\sin u}_{r.} \sin v, a\cos u \rangle$$

$0 \le u \le \pi$ $\qquad$ $0 \le v. \le 2\pi$

$v = \theta.$

$0 \le \theta \le 2\pi.$

$0 \le u \le \frac{\pi}{2}$

$\frac{\pi}{2} \le u \le \pi.$

④ paraboloid.

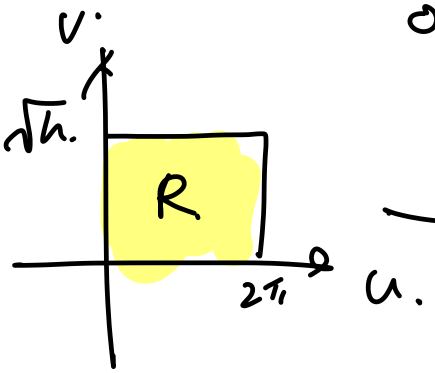
$x^2+y^2=z$ $\boxed{0 \le z < h}$

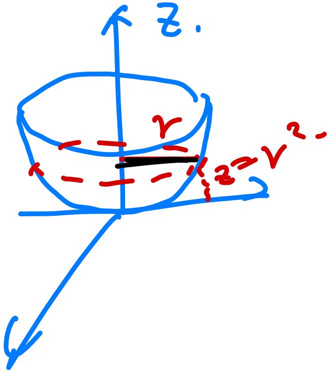
$$\boxed{r(u.v). = \langle v\cos u,\ v\sin u,\ v^2 \rangle}$$

$0 \le u \le 2\pi$

$0 \le v^2 \le h.$

$0 \le v \le \sqrt{h}.$





$x = r\cos\theta$

$y = r\sin\theta.$

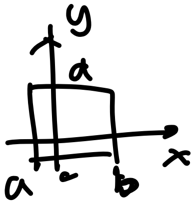
$z = x^2+y^2$

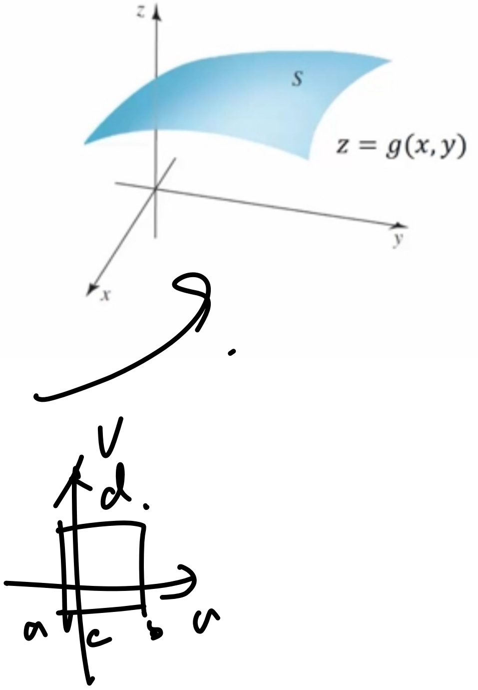
$= r^2.$

$\theta = u.$ $r = V.$

⑤ $\underline{z = g(x,y)}$ ⇐ Explicitly defined surface.

$R = \{(x,y).\ a \le x \le b,\ c \le y \le d.\}$





$\boxed{u=x}$ $\boxed{v=y}$

$$\boxed{\begin{gathered} r(u,v) = \langle \underline{u},\ \underline{v},\ \underline{g(u,v)} \rangle, \\ a \le u \le b \qquad 0 \le v \le d. \end{gathered}}$$

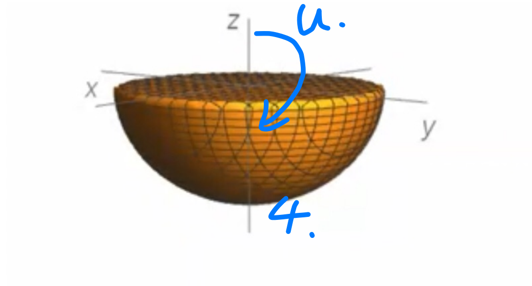
$\boxed{\text{ex).}}$ hemisphere $a^2=16$

a. $x^2+y^2+z^2=\boxed{16.}$ for $z \le 0$.

How to write $r(u,v)$?

$r(u,v) = \langle a.\sin u\cos v, a\sin u\sin v, a\cos u\rangle.$

$0 \le u \le \pi \qquad 0 \le v \le 2\pi$



radius $a=4$.

$\frac{\pi}{2} \le u \le \pi.$

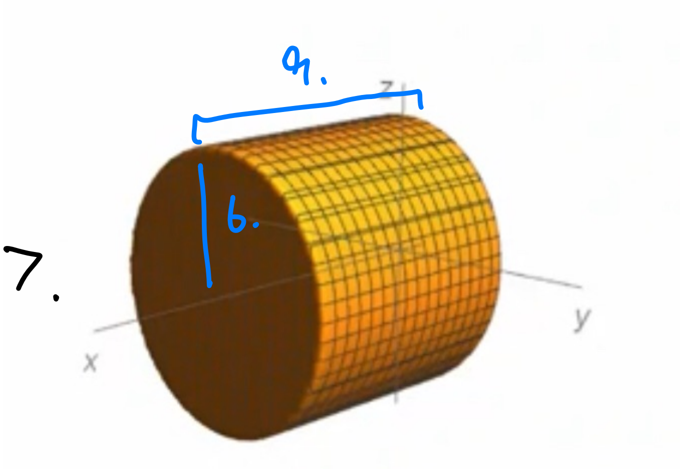
$$\boxed{\begin{array}{c} r(u.v) = \langle 4\sin u \cos v, 4\sin u.\cos v, a\cos u\rangle. \\ \frac{\pi}{2} \le u \le \pi \qquad 0 \le v \le 2\pi \end{array}}$$

ex2. The cylinder $y^2+z^2=36$. , $0 \le x \le 9$.

radius $a^2=36$. $a=6$.

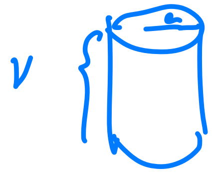
height $h=9$.

axis along x-axis



$\vec{r}(u,v)=\langle v, 6\cos u, 6\sin u\rangle$.

$0 \le u \le 2\pi$. $0 \le v \le 9$.



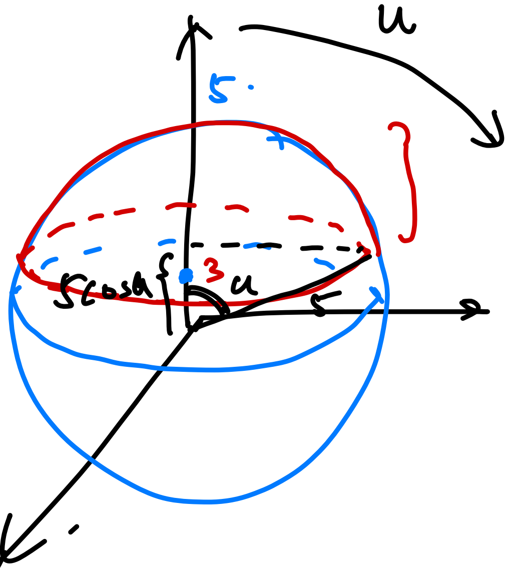
axis - z-axis

$r(u,v)$

$=\langle a\cos u, b\sin u, v\rangle$

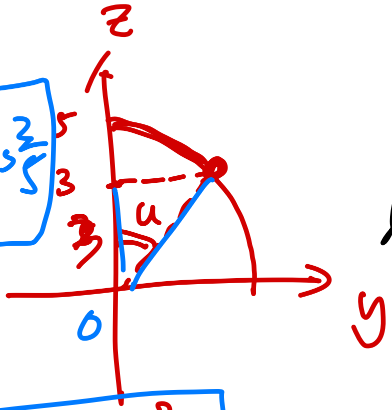
ex). sphere. radius 5.
centered at origin.
from top to heigh 3.

$r(u,v) = \langle 5\sin u\cos v, 5\sin u\sin v, 5\cos u \rangle$.



$0 \le v \le 2\pi$

$0 \le \boxed{u\,?} \le \arccos\frac{3}{5}$



$\boxed{\cos u = \frac{3}{5}.}$

$u = \arccos\frac{3}{5}.$

ex) paraboloid.

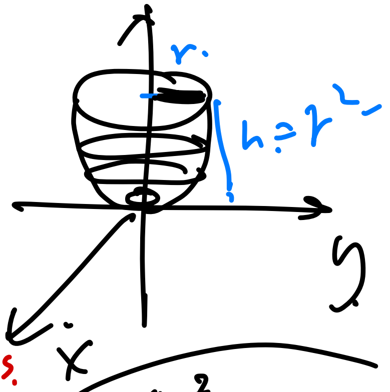
radius. 5 (r=5)

height 25.

$\boxed{z = x^2+y^2. \quad 0 \le z \le h.}$

$x = r\cos\theta \quad y = r\sin\theta$

$z = x^2+y^2 = r^2.$



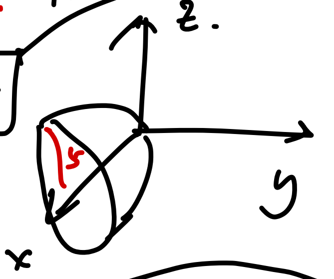
$\vec{r}(u,v) = \langle v\cos u, v\sin u, \boxed{v^2} \rangle.$ → axis is z-axis.

$0 \le u \le 2\pi \quad 0 \le v \le 5.$

radius 5.

height 25.

$\boxed{x = z^2+y^2. \quad 0 \le x \le h.}$



$\vec{r}(u,v) = \langle v^2, v\cos u, v\sin u \rangle.$

$0 \le u \le 2\pi \quad 0 \le v \le 5.$

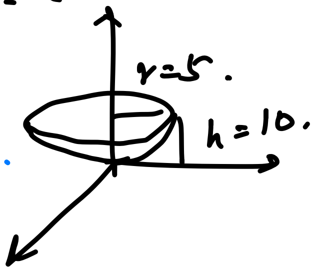
radius ⑤

height ⑩

$z = a(x^2+y^2) \quad 0 \le z \le h$

$x = 5\cos\theta \quad y = 5\sin\theta.$

$10 = z = a \cdot 5^2 \Rightarrow a = \frac{10}{25}.$



$\vec{r}(u,v) = \langle v\cos u, v\sin u, \boxed{\frac{10}{25}v^2} \rangle$

$0 \le u \le 2\pi \quad 0 \le v \le 5.$

radius = v.

$z = a \cdot v^2 = \frac{10}{25}v^2.$

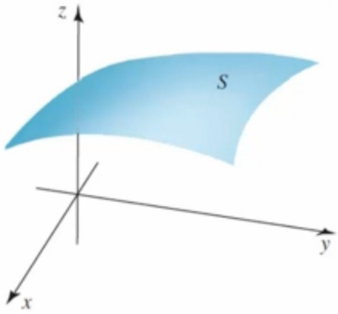
$v=5 \Rightarrow h = \frac{10}{25} \cdot v^2 = \frac{10}{25} \cdot 25 = 10.$

Surface Integrals of Scalar-Valued Functions

Next: We will develop the surface integral of a scalar-valued function f defined on a smooth parameterized surface S.

$$\iint\limits_S f(x,y,z)\,\boxed{dS}$$

Region of integration



Applications:

- Compute the surface area of S. $f = 1.$
- Compute the mass of a thin sheet described by the surface S with mass density function f.
- Compute the average value of f over the surface S.

Surface Integrals of Scalar-Valued Functions

Definition: Let f be a continuous scalar-valued function on a smooth surface S given parametrically by $\boxed{\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle}$, where u and v vary over the rectangle $R = \{(u,v): a \le u \le b, c \le v \le d\}$. Assume also that the tangent vectors

$$\underline{\mathbf{t}_u = \boxed{\frac{\partial \mathbf{r}}{\partial u}} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle} \qquad \text{and} \qquad \underline{\mathbf{t}_v = \boxed{\frac{\partial \mathbf{r}}{\partial v}} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle}$$

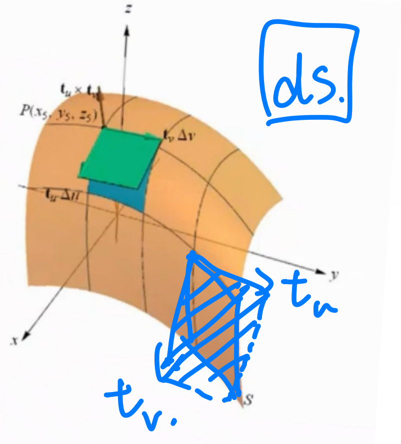
are continuous on R and the normal vector $\boxed{\mathbf{t}_u \times \mathbf{t}_v}$ is nonzero on R.

The **surface integral of f over S** is

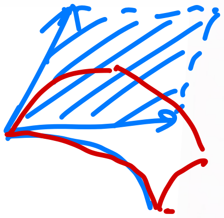
$$\iint_S f(x,y,z)\,dS = \iint_R f\big(x(u,v), y(u,v), z(u,v)\big) \boxed{|\mathbf{t}_u \times \mathbf{t}_v|}\, dA.$$

$\boxed{ds.}$

If $f(x,y,z) = 1$, this integral equals the surface area of S.



$\boxed{\|t_u \times t_v\|} \approx$ area of ds.

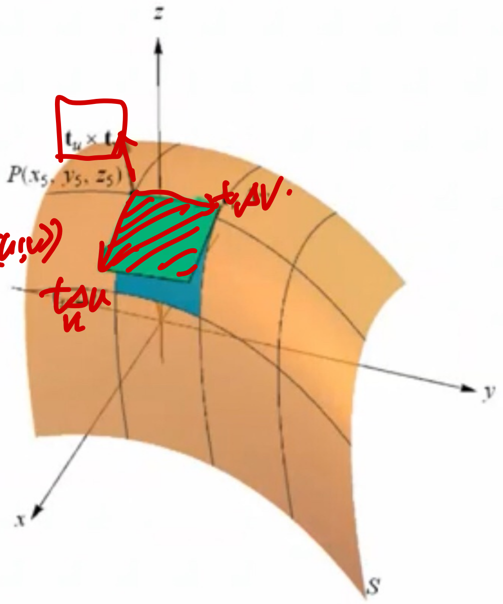


Surface Integrals of Scalar-Valued Functions

$$\iint_S f(x, y, z)\,\boxed{dS} =$$

$$\iint_R f\big(x(u,v), y(u,v), z(u,v)\big)\,|\mathbf{t}_u \times \mathbf{t}_v|\,dA$$

$\vec{r}(u.v) = x(u.v), y(u.v), z(u,v)$



Interpretation:

- Partition the surface S into subregions S_k corresponding to rectangles in the uv-plane.
- Add up the values $\underline{f(x_k, y_k, z_k)\Delta S_k}$ to approximate the surface integral.
- $|\mathbf{t}_u \times \mathbf{t}_v|$ is the area of the parallelogram formed by the tangent vectors.
- The area of $S_k = \Delta S_k \approx \underline{|\mathbf{t}_u \times \mathbf{t}_v|}\,\Delta u\,\Delta v$.

S: $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$

$R: \{(u,v), a \leq u \leq b, c \leq v \leq d\}.$

$z = g(x, y).$

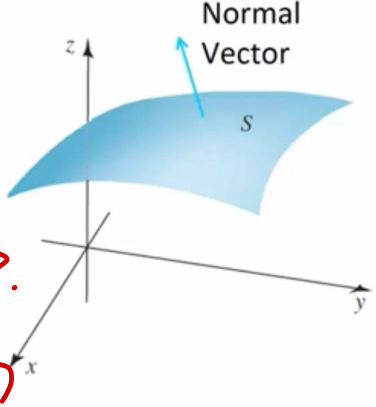
Surface Integrals of Scalar-Valued Functions

Example: Explicitly Defined Surface $\underline{z = g(x,y)}$ on $\underline{R = \{(x,y): a \le x \le b, c \le y \le d\}}$.

Parametric Description: $\boxed{\mathbf{r}(x,y) = \langle x, y, g(x,y)\rangle,}$
$\underline{\text{where } a \le x \le b \text{ and } c \le y \le d}$

$x = u$
$y = v.$
$z = g(x,y) = g(u.v)$

Tangent Vectors: $\mathbf{t}_x = \langle 1, 0, z_x\rangle$ and $\mathbf{t}_y = \langle 0, 1, z_y\rangle$



Normal Vector:

$$\boxed{\mathbf{t}_x \times \mathbf{t}_y = \langle -z_x, -z_y, 1\rangle}$$

$$t_x = \frac{\partial r}{\partial x} = \langle 1, 0, \frac{\partial g}{\partial x} \rangle = \langle 1, 0. z_x \rangle.$$

$$t_y = \frac{\partial r}{\partial y} = \langle 0, 1, \frac{\partial g}{\partial y} \rangle = \langle 0, 1, z_y \rangle.$$

$$t_x \times t_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{vmatrix} = \begin{vmatrix} 0 & z_x \\ 1 & z_y \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & z_x \\ 0 & z_y \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \vec{k}$$

$$= (-z_x)\vec{i} - (z_y)\vec{j} + 1\vec{k}$$

$$= \langle -z_x, -z_y, 1 \rangle.$$

$$\|t_x \times t_y\| = \sqrt{z_x^2 + z_y^2 + 1}$$

$$\iint_S f(x.y\,z)\,dS = \iint_R f(x.y.z) \overbrace{\sqrt{z_x^2 + z_y^2 + 1}}^{\|t_u \times t_v\|} \overset{dxdy}{\underline{dA}}.$$

Evaluation of Surface Integrals of Scalar-Valued Functions on Explicitly Defined Surfaces

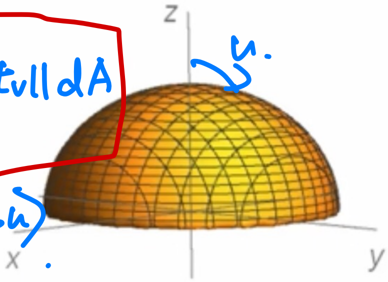
Theorem: Let f be a continuous scalar-valued function on a smooth surface S given parametrically by $z = g(x,y)$, for (x,y) in a region R. The **surface integral of f over S** is

$$\iint_S f(x,y,z)\,dS = \iint_R f\big(x,y,g(x,y)\big)\,\underbrace{\sqrt{z_x^2 + z_y^2 + 1}}_{\left|\mathbf{t}_x \times \mathbf{t}_y\right|}\,dA.$$

If $f(x,y,z) = 1$, the surface integral equals the area of the surface.

Example: Find the surface area of the hemisphere $x^2 + y^2 + z^2 = 100$ for $\boxed{z \geq 0}$ using a parametric description of the surface.

$10^2.$



Surface Area $= \iint_S (1)\,dS = \boxed{\iint_R 1\,\|t_u \times t_v\|\,dA}$

$\vec{r}(u,v) = \langle 10\sin u\cos v, 10\sin u\sin v, 10\cos u\rangle$

$0 \leq u \leq \frac{\pi}{2}$ $0 \leq v \leq 2\pi$

$t_u = \frac{\partial r}{\partial u} = \langle 10\cos u\cos v, 10\cos u\sin v$

$\langle 0\ -\sin v\rangle$

$\|\vec{t}_u \times \vec{t}_v\| = \boxed{10^2\sin u}$

	Explicit Description $z = g(x,y)$		Parametric Description	
Surface	**Equation**	**Normal vector; magnitude** $\pm\langle -z_x, -z_y, 1\rangle; \lvert\langle -z_x, -z_y, 1\rangle\rvert$	**Equation**	**Normal vector; magnitude** $t_u \times t_v; \boxed{\lvert t_u \times t_v\rvert}$
Cylinder	$x^2 + y^2 = a^2$, $0 \leq z \leq h$	$\langle x, y, 0\rangle; a$	$\mathbf{r} = \langle a\cos u, a\sin u, v\rangle$, $0 \leq u \leq 2\pi, 0 \leq v \leq h$	$\langle a\cos u, a\sin u, 0\rangle; \boxed{a}$
Cone	$z^2 = x^2 + y^2$, $0 \leq z \leq h$	$\langle x/z, y/z, -1\rangle; \sqrt{2}$	$\underline{\mathbf{r} = \langle v\cos u, v\sin u, v\rangle,}$ $0 \leq u \leq 2\pi, 0 \leq v \leq h$	$\langle v\cos u, v\sin u, -v\rangle; \boxed{\sqrt{2}v}$
$\boxed{\textbf{Sphere}}$	$\underline{x^2 + y^2 + z^2 = a^2}$	$\langle x/z, y/z, 1\rangle; a/z$	$\mathbf{r} = \langle a\sin u\cos v$, $a\sin u\sin v, a\cos u\rangle$, $0 \leq u \leq \pi, 0 \leq v \leq 2\pi$	$\langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v$, $a^2\sin u\cos u\rangle; \boxed{a^2\sin u}$
Paraboloid	$z = x^2 + y^2$, $0 \leq z \leq h$	$\langle 2x, 2y, -1\rangle; \sqrt{1 + 4(x^2 + y^2)}$	$\mathbf{r} = \langle v\cos u, v\sin u, v^2\rangle$, $0 \leq u \leq 2\pi, 0 \leq v \leq \sqrt{h}$	$\langle 2v^2\cos u, 2v^2\sin u, -v\rangle; \boxed{v\sqrt{1 + 4v^2}}$

$$\iint_R \|t_u \times t_v\| \, du \, dv. = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} 1 \cdot 10^2 \sin u \, du \, dv.$$

$0 \le u \le \frac{\pi}{2}.$

$0 \le v \le 2\pi.$

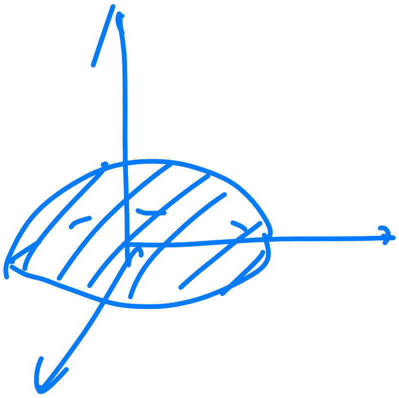
$$= \int_0^{2\pi} \boxed{\int_0^{\frac{\pi}{2}} 100 \sin u \quad du} \, dv.$$

$$= \int_0^{2\pi} -100 \cos u \Big|_0^{\frac{\pi}{2}} \quad dv$$

$$= \int_0^{2\pi} \left[\cancel{-100 \cos \tfrac{\pi}{2}}. + 100 \cos 0\right] dv$$

$$= \int_0^{2\pi} 100 \, dv$$

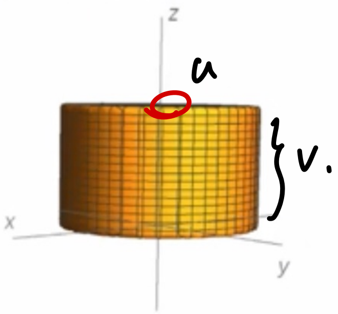
$$= \boxed{200\pi}$$



Example: Evaluate the given surface integral using a parametric description of the surface S; the cylinder $x^2 + y^2 = 9$ for $0 \leq z \leq 3$.

3^2

$$\boxed{\iint_S y\, dS}$$



radius = 3. height = 3.

$$\boxed{\vec{r}(u,v) = \langle 3\cos u, 3\sin u, v \rangle.}$$ for $0 \leq u \leq 2\pi$

$0 \leq v \leq 3.$

$$|\vec{t}_u \times \vec{t}_v| = 3 \Rightarrow \iint_S y\, dS = \int_0^{2\pi}\int_0^3 \underbrace{3\sin u}_{f=y} \cdot \underbrace{\boxed{3}}_{|\vec{t}_u \times \vec{t}_v|} dv\, du$$

$= 3\sin u.$

$$\iint_S f\, dS = \iint_R f(x(u,v), y(u,v), z(u,v))\, |\vec{t}_u \times \vec{t}_v|\, dA.$$

$$= \int_0^{2\pi} 9\sin u \cdot v \Big|_{v=0}^{v=3} du$$

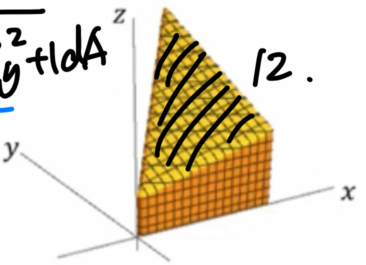
$$= \int_0^{2\pi} 27\sin u\, du$$

$$= -27\cos u \Big|_0^{2\pi} = -27 - (-27) = 0.$$

	Explicit Description $z = g(x, y)$		**Parametric Description**	
Surface	**Equation**	**Normal vector; magnitude** $\pm\langle -z_x, -z_y, 1\rangle; \lvert\langle -z_x, -z_y, 1\rangle\rvert$	**Equation**	**Normal vector; magnitude** $t_u \times t_v; \boxed{\lvert t_u \times t_v\rvert}$
Cylinder	$x^2 + y^2 = a^2$, $0 \le z \le h$	$\langle x, y, 0\rangle; a$	$\mathbf{r} = \langle a\cos u, a\sin u, v\rangle$, $0 \le u \le 2\pi, 0 \le v \le h$	$\langle a\cos u, a\sin u, 0\rangle; \boxed{a}$
Cone	$z^2 = x^2 + y^2$, $\underline{0 \le z \le h}$	$\underline{\langle x/z, y/z, -1\rangle}; \sqrt{2}$	$\mathbf{r} = \langle v\cos u, v\sin u, v\rangle$, $0 \le u \le 2\pi, 0 \le v \le h$	$\boxed{\langle v\cos u, v\sin u, -v\rangle}; \boxed{\sqrt{2}v}$
Sphere	$x^2 + y^2 + z^2 = a^2$	$\langle x/z, y/z, 1\rangle; a/z$	$\mathbf{r} = \langle a\sin u\cos v$, $a\sin u\sin v, a\cos u\rangle$, $0 \le u \le \pi, 0 \le v \le 2\pi$	$\langle a^2\sin^2 u\cos v, \boxed{a^2\sin^2 u\sin v,}$ $a^2\sin u\cos u\rangle; \boxed{a^2\sin u}$
Paraboloid	$z = x^2 + y^2$, $0 \le z \le h$	$\langle 2x, 2y, -1\rangle; \sqrt{1 + 4(x^2 + y^2)}$	$\mathbf{r} = \langle v\cos u, v\sin u, v^2\rangle$, $0 \le u \le 2\pi, 0 \le v \le \sqrt{h}$	$\langle 2v^2\cos u, 2v^2\sin u, -v\rangle; \boxed{v\sqrt{1 + 4v^2}}$

Example: Find the area of part of the plane $z = 2x + 2y + 4$ over the region R bounded by the triangle with vertices $(0, 0)$, $(2, 0)$, and $(2, 4)$ using an explicit description of the surface.

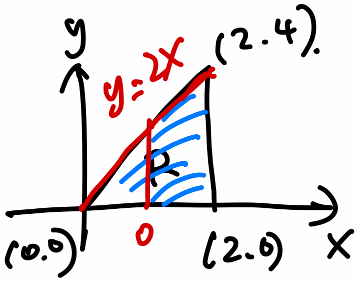
use Surface Area $= \iint_S \boxed{(1)}\, dS = \iint_R (1)\sqrt{z_x^2 + z_y^2 + 1}\, dA$



12.

$z = 2x + 2y + 4 \Rightarrow z_x = 2.$ and $z_y = 2.$

$R: \{(x, y): \quad 0 \le x \le 2, \quad 0 \le y \le 2x\}.$



$SA = \iint_{0\ 0}^{2\ 2x} 1 \cdot \sqrt{2^2 + 2^2 + 1}\, dy\, dx$

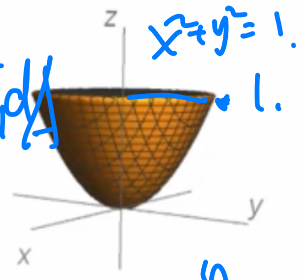
$= \int_0^2 \int_0^{2x} \sqrt{9}\, dy\, dx$

$= 3 \int_0^2 y \Big|_0^{2x} dx$

$= 3 \int_0^2 2x\, dx = 3 \cdot x^2 \Big|_0^2 = 3(2^2 - 0) = 3 \cdot 4 = 12.$

Example: Evaluate the given surface integral using an explicit description of the surface S; the paraboloid $z = x^2 + y^2$ for $0 \le z \le 1$.

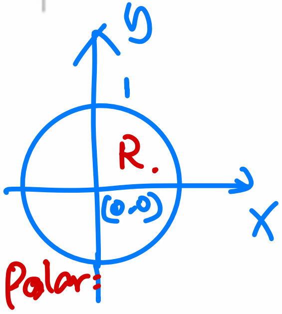
$$\iint_S (x^2 + y^2)\, dS$$



Use $\iint_S f(x,y,z)\,ds = \iint_{\boxed{R}} f(x,y,g(x,y))\sqrt{z_x^2+z_y^2+1}\,dA$

$z = x^2+y^2 \Rightarrow z_x = 2x$ and $z_y = 2y$.

$$\iint_S (x^2+y^2)\,dS = \iint_R (x^2+y^2)\underbrace{\sqrt{(2x)^2+(2y)^2+1}}_{2(x^2+y^2)=4r^2.}\ \boxed{dA}$$



Use Polar:

$R = \{(r,\theta),\ 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

$$= \int_0^{2\pi}\int_0^1 r^2\sqrt{4r^2+1}\ r\,dr\,d\theta$$

$$= \int_0^1\int_0^{2\pi} r^2\sqrt{4r^2+1}\ r\,d\theta\,dr.$$

$u = 4r^2+1$

$du = 8r\,dr.$ $r\,dr = \frac{1}{8}du$ $r^2 = \frac{u-1}{4}.$

$r=0$ $u=1$

$r=1$ $u=5$

$$= 2\pi\int_0^1 r^2\sqrt{4r^2+1}\ \boxed{r\,dr} = 2\pi\int^5 \frac{u-1}{4}\sqrt{u}\ \frac{1}{8}du.$$

$$= 2\pi\int_1^5 \frac{1}{32}\left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$$

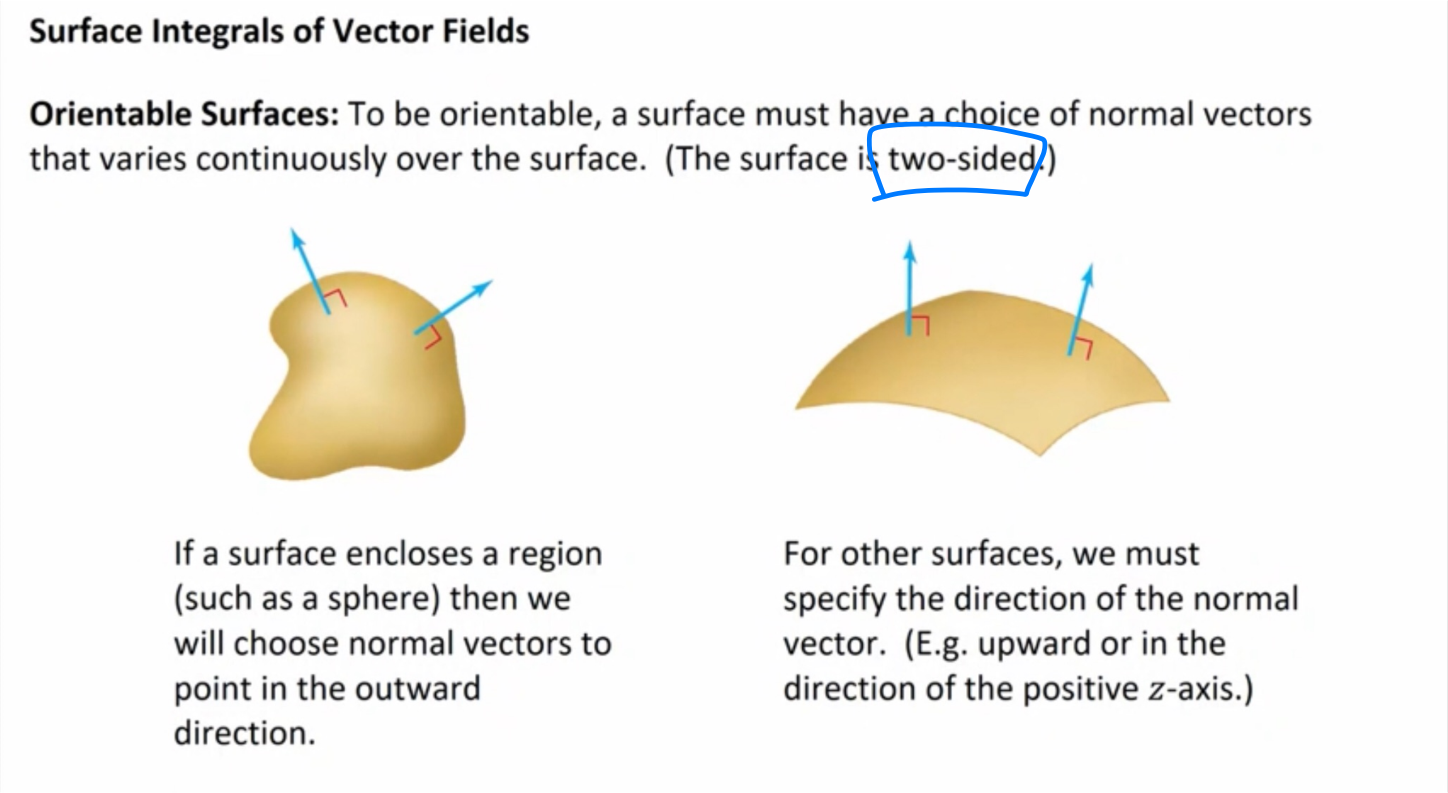
$$\frac{2\pi}{32}\int_1^5 u^{\frac{3}{2}} - u^{\frac{1}{2}} du$$

$$= \frac{\pi}{16}\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]\Bigg|_{u=1}^{u=5}$$

$$= \frac{\pi}{16}\left[\frac{2}{5}(\sqrt{5})^5 - \frac{2}{3}(\sqrt{5})^3 - \frac{2}{5} + \frac{2}{3}\right]$$

$$= \frac{\pi}{16}\left[\frac{2}{5}(\sqrt{5})^4 \cdot \sqrt{5} - \frac{2}{3}(\sqrt{5})^2\sqrt{5} - \frac{-4}{15}\right]$$

$$= \frac{\pi}{16}\left[\frac{2}{5}\cdot 25\sqrt{5} - \frac{2}{3}5\sqrt{5} + \frac{4}{15}\right]$$



Surface Integrals of Vector Fields

Orientable Surfaces: To be orientable, a surface must have a choice of normal vectors that varies continuously over the surface. (The surface is two-sided.)

If a surface encloses a region (such as a sphere) then we will choose normal vectors to point in the outward direction.

For other surfaces, we must specify the direction of the normal vector. (E.g. upward or in the direction of the positive z-axis.)

Surface Integrals of Vector Fields

Flux Integrals: Consider a continuous vector field $\mathbf{F} = \langle f, g, h \rangle$. Let S be a smooth oriented surface with unit normal vector $\mathbf{n}$.

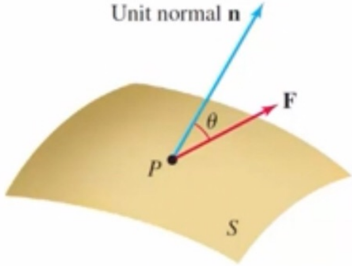
Definition: The flux integral

$$\iint_S \boxed{\mathbf{F} \cdot \mathbf{n}} \, dS = \iint_R F \cdot (t_u \times t_v)\, dA.$$

Surface Integral of Scalar using $|t_u \times t_v|$ but we take the dot product of F with $t_u \times t_v$.

computes the net flux of the vector field across the surface.

rate of flow of vector field through S.



$$\boxed{\mathbf{F} \cdot \mathbf{n}} = |\mathbf{F}||\mathbf{n}| \cos\theta = |\mathbf{F}| \cos\theta$$

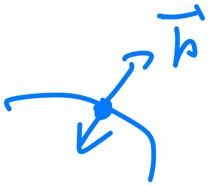
The flux integral adds up the components of the vector field $\mathbf{F}$ normal to the surface.

Surface Integrals of Vector Fields

Definition: Surface Integral of a Vector Field

Suppose $\mathbf{F} = \langle f, g, h \rangle$ is a continuous vector field on a region of $\mathbb{R}^3$ containing a smooth oriented surface S. If S is defined parametrically as $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$, for (u,v) in a region R, then

$$\iint_S \mathbf{F} \cdot \boxed{\mathbf{n}\, dS} = \iint_R \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v)\, dA.$$



where $\mathbf{t}_u = \dfrac{\partial \mathbf{r}}{\partial u} = \left\langle \dfrac{\partial x}{\partial u}, \dfrac{\partial y}{\partial u}, \dfrac{\partial z}{\partial u} \right\rangle$ and $\mathbf{t}_v = \dfrac{\partial \mathbf{r}}{\partial v} = \left\langle \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial v}, \dfrac{\partial z}{\partial v} \right\rangle$ are continuous on R,

and the normal vector $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R, and the direction of the normal vector is consistent with the orientation of S.

Important to check! If not consistent, change the sign.

why?. $\vec{n} = \dfrac{\vec{t}_u \times \vec{t}_v}{\cancel{|\vec{t}_u \times \vec{t}_v|}}$ $dS = \cancel{|\vec{t}_u \times \vec{t}_v|}\, dA$.

$\vec{n}\, dS = \vec{t}_u \times \vec{t}_v \; dA$.

Surface Integrals of Vector Fields

Definition: Surface Integral of a Vector Field

If S is defined in the form $\boxed{z = w(x,y)}$ for (x,y) in a region R, then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \mathbf{F}\cdot\underline{(\mathbf{t}_u\times\mathbf{t}_v)}\,dA = \iint_R \boxed{(-fz_x - gz_y + \boxed{h})}\,dA.$$

Alway compute the upward flux.

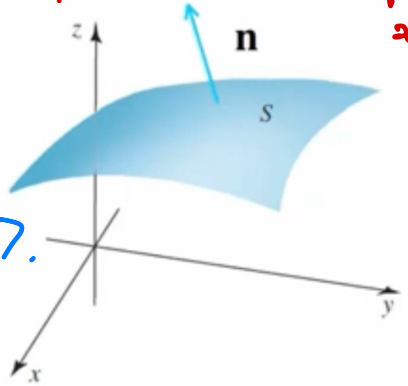
$\because$ positive zaxis.

① $F = \langle f, g, h \rangle$.

② $t_u \times t_v = \langle -z_x, -z_y, \boxed{1} \rangle$. for surface S $z = w(x,y)\rangle$.

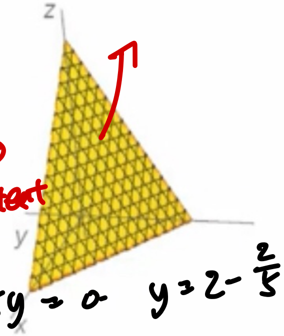
Thus $F\cdot(t_u \times t_v) = \langle f, g, h \rangle \cdot \langle -z_x, -z_y, 1 \rangle$.

$= \boxed{-f z_x - g z_y + h.}$



Example: Find the upward flux of the vector field $\mathbf{F} = \langle x, y, z\rangle$ across the slanted face of the tetrahedron $z = 10 - 2x - 5y$ in the first octant. You may use either an explicit or a parametric description of the surface. $z = w(x, y)$.

use $\iint_S F \cdot n\,ds. = \iint_{\boxed{R}} (-f z_x - g z_y + h)\,dA.$



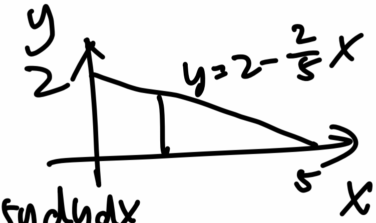
checked :

normal $t_u \times t_v = \langle -\underline{z_x}, -\underline{z_y}, 1\rangle = \langle 2, 5, \boxed{1}\rangle.$ upward → consistent

$z = 10 - 2x - 5y \Rightarrow z_x = -2$ and $z_y = -5.$

$z = 0.$

$10 - 2x - 5y = 0 \quad y = 2 - \frac{2}{5}x$



$$\iint_S F \cdot \vec{n}\,ds = \iint_R 2x + 5y + z\,dA$$

$$= \int_0^5 \int_0^{2-\frac{2}{5}x} 2x + 5y + z\,dy\,dx = \int_0^5 \int_0^{2-\frac{2}{5}x} 2x + 5y + \underbrace{10 - 2x - 5y}_{z}\,dy\,dx$$

$$= \int_0^5 \int_0^{2-\frac{2}{5}x} 10\,dy\,dx = \int_0^5 10\left(2 - \frac{2}{5}x\right)dx$$

$$= 20x - \frac{4}{2}x^2 \Big|_0^5 = 20 \cdot 5 - 2 \cdot 25 - 0 = 100 - 50 = 50$$

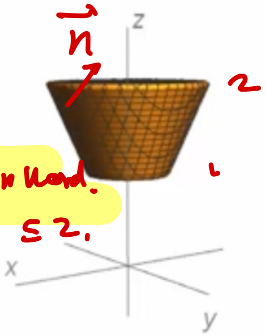
$2 - \frac{2}{5}x = 0$

$2 = \frac{2}{5}x$

$x = 5$

$R = \{(x, y).\ 0 \le x \le 5.\ 0 \le y \le 2 - \frac{2}{5}x\}$

Example: Find the upward flux of the vector field $\mathbf{F} = \langle -x, -y, z^2 \rangle$ across the portion of the cone $z = \sqrt{x^2 + y^2}$ between $z = 1$ and $z = 2$. You may use either an explicit or a parametric description of the surface.



Use $\iint_S F \cdot \vec{n}\, ds = \iint_R F \cdot (t_u \times t_v)\, dA$

cone: $t_u \times t_v = \langle v\cos u, v\sin u, \boxed{-v} \rangle$ ← point downward. for $1 \le v \le 2$.

radius = height

$$\boxed{t_u \times t_v = \langle -v\cos u, -v\sin u, v \rangle.}$$

$0 \le u \le 2\pi$. $1 \le v < 2$.

$\vec{r}(u,v) = \langle v\cos u, v\sin u, v \rangle$

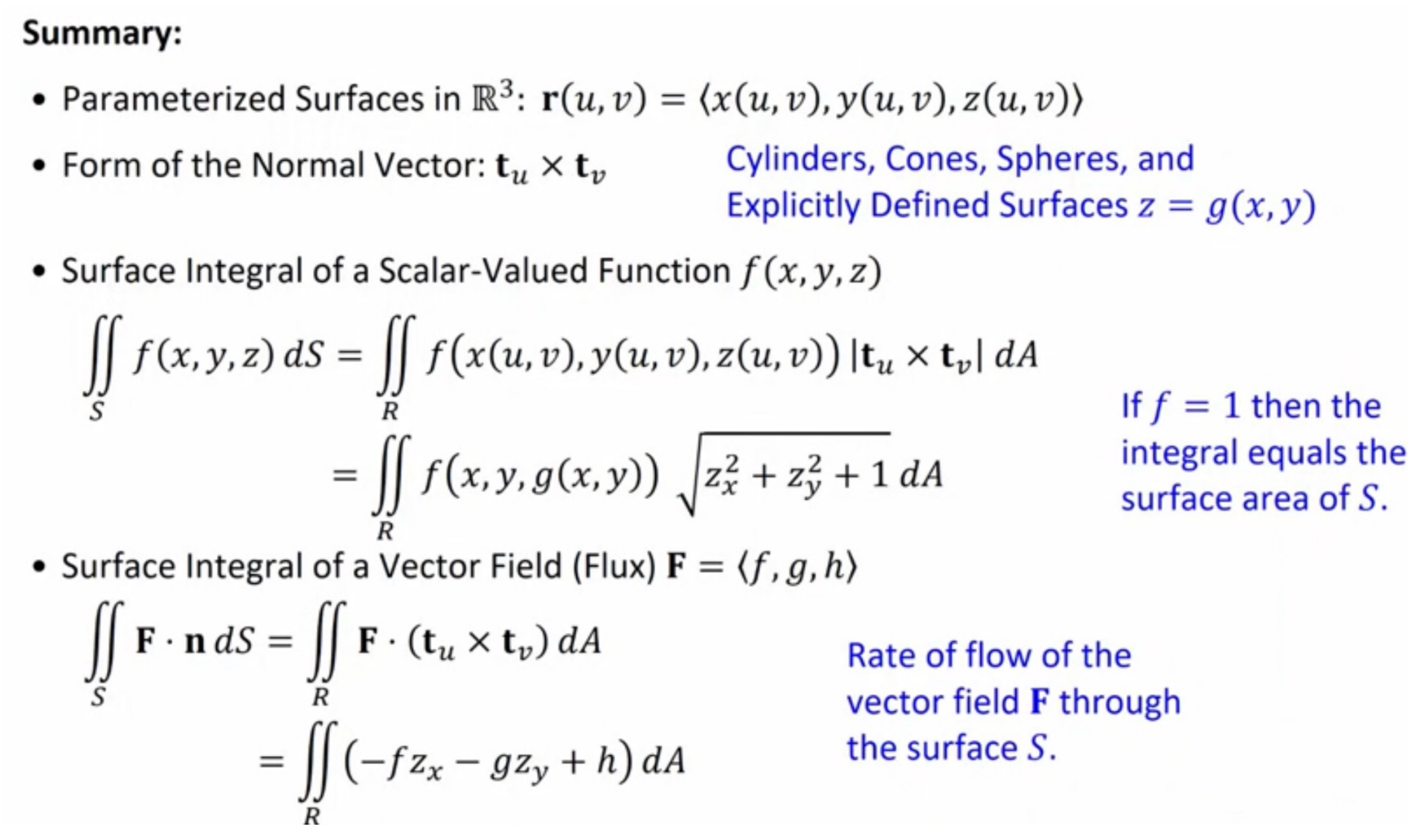
$\vec{F} = \langle -x, -y, z^2 \rangle$.

$\vec{F} \cdot (t_u \times t_v) = \langle -v\cos u, -v\sin u, v^2 \rangle \cdot \langle -v\cos u, -v\sin u, v \rangle$

$= v^2\cos^2 u + v^2\sin^2 u + v^3 = v^2 + v^3$.

$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_R v^2 + v^3\, dA = \int_1^2\int_0^{2\pi} v^2 + v^3\, du\, dv = 2\pi \int_1^2 v^2 + v^3\, dv = 2\pi\left(\frac{1}{3}v^3 + \frac{1}{4}v^4\right)\Big|_1^2$

$= \frac{73\pi}{6}$ ← upward flux



Summary:

- Parameterized Surfaces in $\mathbb{R}^3$: $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$
- Form of the Normal Vector: $\mathbf{t}_u \times \mathbf{t}_v$

Cylinders, Cones, Spheres, and Explicitly Defined Surfaces $z = g(x,y)$

- Surface Integral of a Scalar-Valued Function $f(x,y,z)$

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_R f\big(x(u,v), y(u,v), z(u,v)\big)\,|\mathbf{t}_u \times \mathbf{t}_v|\,dA$$

$$= \iint\limits_R f\big(x, y, g(x,y)\big)\,\sqrt{z_x^2 + z_y^2 + 1}\,dA$$

If $f = 1$ then the integral equals the surface area of S.

- Surface Integral of a Vector Field (Flux) $\mathbf{F} = \langle f, g, h\rangle$

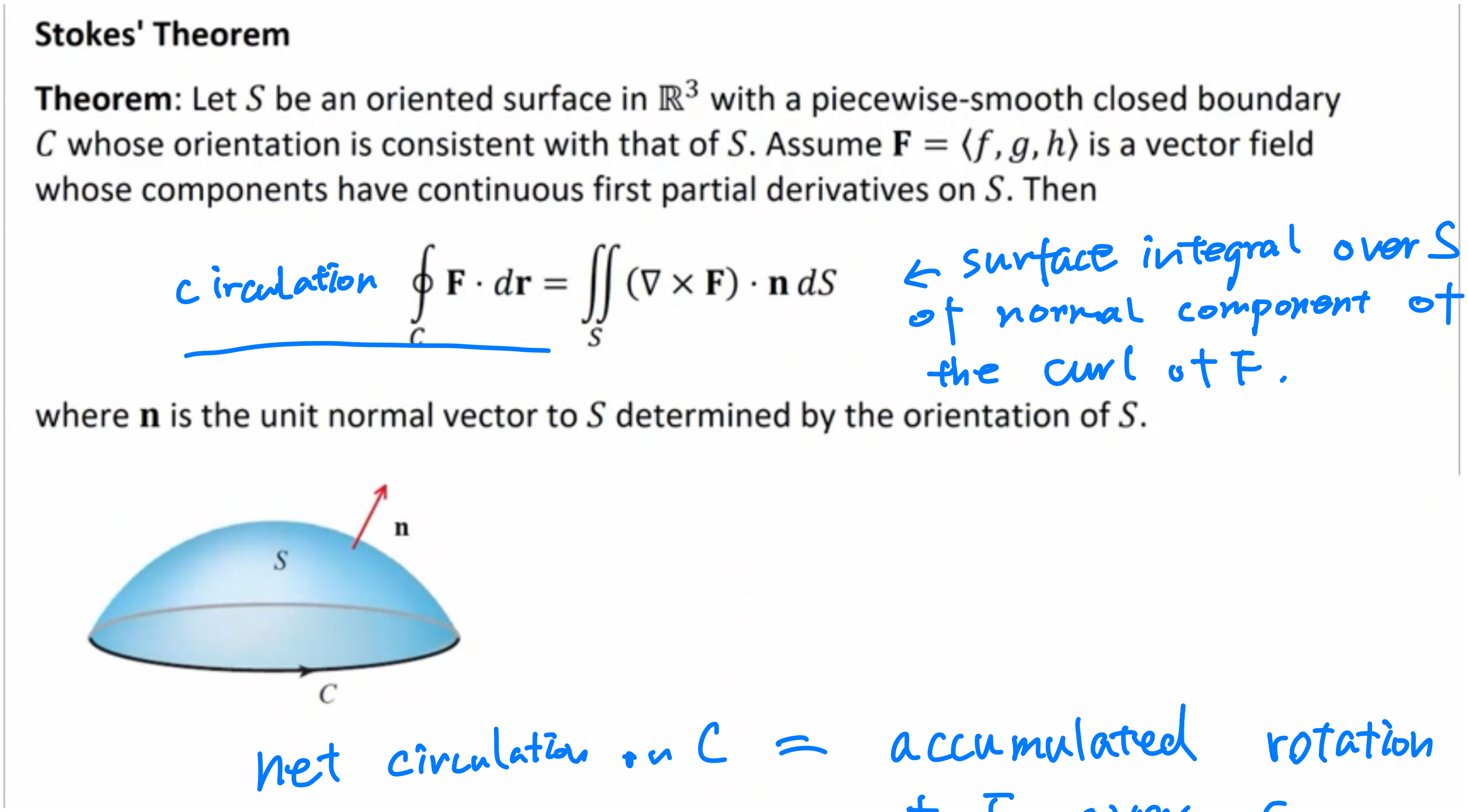
$$\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint\limits_R \mathbf{F}\cdot(\mathbf{t}_u \times \mathbf{t}_v)\,dA$$

$$= \iint\limits_R \big(-f z_x - g z_y + h\big)\,dA$$

Rate of flow of the vector field $\mathbf{F}$ through the surface S.

12.7.

Stokes' Theorem



Stokes' Theorem

Theorem: Let S be an oriented surface in $\mathbb{R}^3$ with a piecewise-smooth closed boundary C whose orientation is consistent with that of S. Assume $\mathbf{F} = \langle f, g, h \rangle$ is a vector field whose components have continuous first partial derivatives on S. Then

$$\underbrace{\text{circulation} \quad \oint_C \mathbf{F} \cdot d\mathbf{r}} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS$$

← surface integral over S of normal component of the curl of F.

where $\mathbf{n}$ is the unit normal vector to S determined by the orientation of S.

net circulation on C = accumulated rotation of F over S.

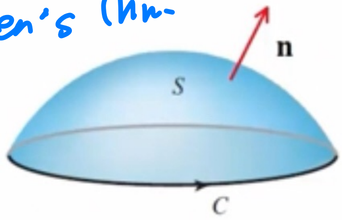
Stokes' Theorem

Stokes' Theorem is the three-dimensional version of the circulation form of Green's Theorem.

3D- Green's Thm.

$$\underline{\oint_C} \mathbf{F}\cdot d\mathbf{r} = \iint_S \underline{(\nabla\times\mathbf{F})\cdot\mathbf{n}}\, dS$$

3D Curl

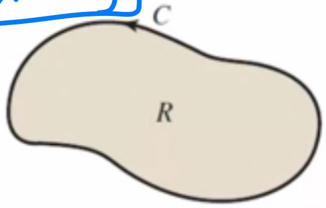


Recall that if C is a closed simple piecewise-smooth oriented curve in the xy-plane enclosing a simply connected region R, and $\mathbf{F} = \langle f, g\rangle$ is a differentiable vector field on R then Green's Theorem says

$$\underline{\oint_C} \mathbf{F}\cdot d\mathbf{r} = \boxed{\iint_R \boxed{(g_x - f_y)}\, dA.}$$

2D Curl.

$$\iint_R \boxed{g_x - f_y}\, dA.$$

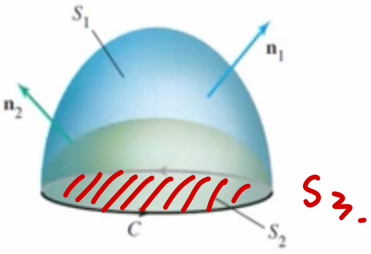


$$\boxed{(-)\iint_R f_y - g_x\, dA} \times .$$

Important Note about Stokes' Theorem

If a closed curve C is the boundary of two different smooth oriented surfaces S_1 and S_2 which both have orientation consisent with that of C, then the integrals of $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$ on the two sufaces are equal.

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_{S_2} (\underline{\nabla \times \mathbf{F}}) \cdot \mathbf{n}\, dS$$



$$\nabla \times F = (\underline{\qquad})\, i + (\underline{\qquad})\, j + (g_x - f_y)\, \vec{k}$$

$$\uparrow \boxed{\vec{n} = \langle 0, 0, 1 \rangle.}$$

$$(\nabla \times F) \cdot \vec{n} = \boxed{g_x - f_y.}$$

$$\iint_{S_3.} \underline{g_x - f_y}\ \underline{dS.}$$

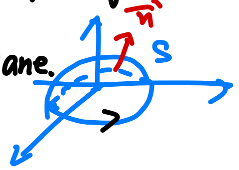
example) ① Verify the Stoke's Thm.

$F = \langle \underset{f}{y}, \underset{g}{-x}, \underset{h}{5} \rangle$ S = upper half of sphere $x^2+y^2+z^2=1$

C = circle $x^2+y^2=1$ on xy-plane.

↳ cc. oriented.

$\vec{n}$ points upward (outward)



$$\boxed{\text{LHS}\ \boxed{\oint_C F\cdot d\vec{r}} \overset{?}{=} \iint_S \text{curl}F\cdot n\, dS \underset{=-2\pi}{}}$$

$$\text{Curl}(F) = \nabla\times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 5 \end{vmatrix} = (0-0)\vec{i} + -(0-0)\vec{j} + \left(\frac{\partial -x}{\partial x} - \frac{\partial y}{\partial y}\right)\vec{k}$$

$$= 0\vec{i} + 0\vec{j} + (-1-1)\vec{k}$$

$$= \langle 0.\ 0,\ -2 \rangle.$$

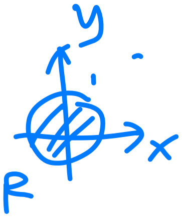
$z = \sqrt{1-x^2-y^2}$

$$\vec{n} = \langle -z_x, -z_y, 1 \rangle$$

$$= \left\langle -\frac{-x}{\sqrt{1-x^2-y^2}}, -\frac{-y}{\sqrt{1-x^2-y^2}}, 1 \right\rangle = \underline{\underline{\left\langle \frac{x}{z}, \frac{y}{z}, 1 \right\rangle}}.$$

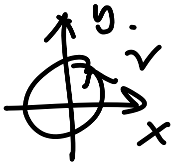
$$\iint_S \text{curF}\cdot\vec{n}\, dS = \iint_R \langle 0,0,-2\rangle \cdot \left\langle \frac{x}{z}, \frac{y}{y}, 1 \right\rangle dA$$

$$= \iint_R -2\, dA. = -2\cdot\pi$$



where R
$\iint_R dA = \pi(1)^2 = \pi.$

The line integral $\oint_C F \cdot \underline{dr}$.



$\vec{r}$ = circle of radius 1. on xy-plane $\boxed{z = 0.}$

$= < \cos t. \sin t, 0 >$, $\quad 0 \le t \le 2\pi$

$\vec{F} = < y. -x, 5 >$. $\boxed{\vec{F}(r(t))} = < \underset{y}{\underline{\sin t}}, \underset{-x}{\underline{-\cos t}}, 5 >$.

$r'(t) = < -\sin t, \cos t, 0 >$.

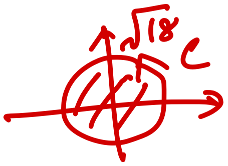
$\oint_C F dr = \oint_0^{2\pi} F \cdot r' dt = \oint_0^{2\pi} \underline{< \sin t, -\cos t, 5 > \cdot < -\sin t, \cos t, 0 > dt}$

$= \oint_0^{2\pi} -\sin^2 t - \cos^2 t + 0 \, dt.$

$= \int_0^{2\pi} -1 \, dt$

$= \boxed{-2\pi.}$ Stoke's Thm ✓.

example 2) Evaluate the line integral $\oint_C F\cdot dr$ by using the surface integral in Stoke's Thm with an appropriate choice of S. Assume that C has cc orientation when view from above.



$F = \langle -9y, -z, x\rangle.$

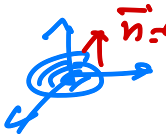
C = circle $x^2+y^2=18.$ on the plane $z=0$

$\oint_C F\cdot dr = ?$

$r(u,v) = \langle u\cos v, u\sin v, 0\rangle.$

$t_u\times t_v = \langle 0.0, u\rangle.$

① surface S: $x^2+y^2=18$ on $z=0$. (Explicit)



$\vec{n} = \langle 0.0.1\rangle.$

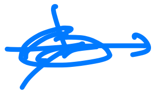
$$\text{curl } F = \nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -9y & -z & x \end{vmatrix} = (0-(-1))\,i + (-1-0)\,\vec{j} + \left(0-\frac{\partial(-9y)}{\partial y}\right)\vec{k}$$

$$= 1\,\vec{i} - 1\,\vec{j} + 9\vec{k}$$

$$\iint_S \text{curl}F\cdot n\,dS = \iint_R \langle 1,-1,9\rangle\cdot\langle 0.0.1\rangle\,dA = \langle 1,-1,9\rangle$$

$$= \iint_R 9\,dA = 9\iint_R dA = 9\pi(\sqrt{18})^2 = 162\pi$$

② Parametric



$r(u,v) = \langle u\cos v,\ u\sin v,\ 0 \rangle$

$0 \le u \le \sqrt{18}$

$0 \le v \le 2\pi$

$t_u \times t_v = \langle 0, 0, u \rangle.$

$\mathrm{curl}(F) = \langle 1, -1, 9 \rangle.$

$$\iint_S \mathrm{curl}F \cdot \vec{n}\, dS = \int_0^{2\pi}\int_0^{\sqrt{18}} \langle 1, -1, 9\rangle \langle 0, 0, u\rangle\, du\, dv.$$

$$= \int_0^{2\pi}\int_0^{\sqrt{18}} 9u\, du\, dv$$

$$= 2\pi \left. \frac{9}{2}u^2 \right|_0^{\sqrt{18}}$$

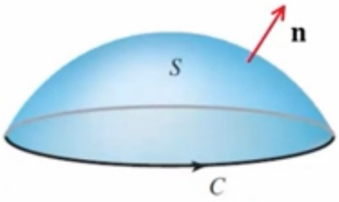
$$= 9\pi(\sqrt{18})^2 = 162\pi.$$

Green's Thm

↓

- **Stokes' Theorem:** Given $\mathbf{F} = \langle f, g, h \rangle$ a vector field in $\mathbb{R}^3$ and S an oriented surface in $\mathbb{R}^3$ with closed boundary C whose orientation is consistent with that of S then under appropriate conditions

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$



↓

$$\oint w = \int dw.$$

where $\mathbf{n}$ is the unit normal vector to S determined by the orientation of S.

- Stokes' Theorem can be used in either direction. That is, it can be used to help evaluate a line integral or to help evaluate a surface integral.

- If C bounds two surfaces then choose the easier one when evaluating the surface integral from Stokes' Theorem!

Divergence Theorem

Divergence Theorem

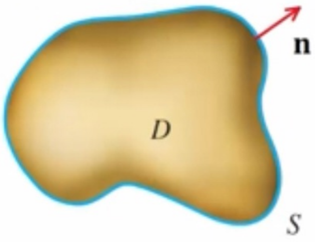
Theorem: Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives in a connected and simply connected region D in $\mathbb{R}^3$ enclosed by an oriented surface S. Then

flux of F across the boundary surface of D.

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \nabla \cdot \mathbf{F}\, dV$$

triple integral over the divergence of F over D.

where $\mathbf{n}$ is the outward unit normal vector to S.



net flow of F across the boundary S.

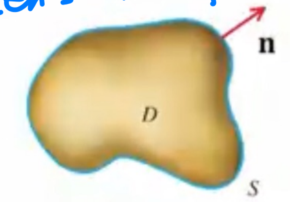
the cumulative expansion or contraction of F over D.

Divergence Theorem

The Divergence Theorem is the three-dimensional version of the flux form of Green's Theorem.

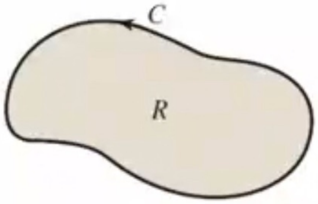
3D - "Green's Th" flux form

$$\underbrace{\boxed{\iint_S \mathbf{F}\cdot\mathbf{n}\,dS}}_{\text{flux}} = \iiint_D \underbrace{\nabla\cdot\mathbf{F}}_{\text{3D-div}}\,dV$$



Recall that if C is a closed simple piecewise-smooth oriented curve in the xy-plane enclosing a simply connected region R, and $\mathbf{F} = \langle f, g\rangle$ is a differentiable vector field on R then Green's Theorem says

$$\underbrace{\boxed{\oint_C \mathbf{F}\cdot\mathbf{n}\,ds}}_{\text{flux}} = \iint_R \underbrace{(f_x + g_y)}_{\text{2D-div}}\,dA$$



ex:) Verify Divergence Thm

$F = \langle 3x, 2y, 3z \rangle \qquad D = \{(x,y,z) \ x^2+y^2+z^2 \le 4\}$

$$\iiint_D \operatorname{div} F \, dV = \iiint_D 3+2+3 \, dV = \iiint_D 8 \, dV = 8 \iiint_D dV = 8(\text{volume of } D)$$

$$= 8\left(\frac{4}{3}\pi 2^3\right) = \frac{2^8}{3}\pi$$

$\iint F \cdot n \, dS$

② Parametric $F = \langle 3x, 2y, 3z \rangle$

$r(u,v) = \langle 2\sin u \cos v, 2\sin u \sin v, 2\cos u \rangle \quad 0 \le u \le \pi$

$\vec{n} = \langle 4\sin^2 u \cos v, 4\sin^2 u \sin v, 4\sin u \cos u \rangle \quad 0 \le v \le 2\pi$

$$\iint_S F \cdot n \, dS = \iint_R \langle 6\sin u \cos v, 4\sin u \sin v, 6\cos u \rangle \cdot \langle 4\sin^2 u \cos v, 4\sin^2 u \sin v, 4\sin u \cos u \rangle \, dA$$

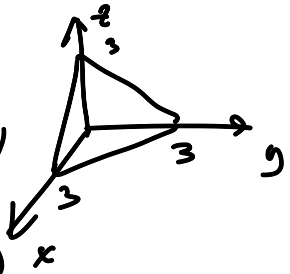
$$= \iint_R 24\sin^3 u \cos^2 v + 16\sin^3 u \sin^2 v + 24\sin u \cos^2 u \, dA$$

... we can integrate this using the trig indentity ... and get $\frac{2^8 \pi}{3}$

ex) Compute the net outward flux on $F = \langle -x, 3y, 2z \rangle$ across S, where S is the boundary of $x+y+z=3$ on the first octend

$$\iint_S F \cdot n \, ds = \iiint_D \nabla \cdot F \, dV = \iiint_D -1+3+2 \, dV = 4 \iiint_D dV$$

$= 4(\text{volume of D})$



what we have is a trirectangular tetrachedron

and the volume is $\frac{abc}{6}$ and in our case $\frac{3 \cdot 3 \cdot 3}{6} = \frac{9}{2}$

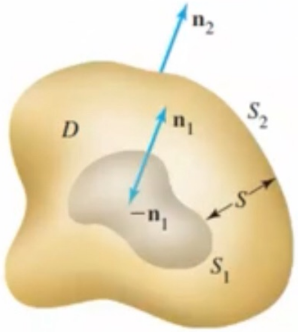
so $\iiint_D 4 \, dV = \frac{9}{2} \cdot 4 = 18.$

If we want to compute the flux on the surface boundary the tretrahedron — compute the flux on 4 surface where you might not want to do.

Divergence Theorem for Hollow Regions

Theorem: Suppose the vector field $\mathbf{F}$ satisfies the conditions of the Divergence Theorem on a region D in $\mathbb{R}^3$ bounded by two oriented surfaces S_1 and S_2 where S_1 lies within S_2. Let S be the entire boundary of D $(S = S_1 \cup S_2)$ and let $\mathbf{n_1}$ and $\mathbf{n_2}$ be the outward unit normal vectors for S_1 and S_2 respectively. Then

$$\iiint\limits_D \underbrace{\nabla \cdot \mathbf{F}}_{\text{divergence}}\, dV = \underbrace{\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS}_{\text{flux}} = \iint\limits_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS - \iint\limits_{S_1} \mathbf{F} \cdot \mathbf{n}_1\, dS.$$



This form of the Divergence Theorem is applicable to vector fields that are not differentiable at the origin, as is the case with some important radial vector fields of the form:

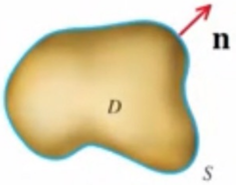
$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p}.$$

Summary

- **Divergence Theorem:** Given $\mathbf{F} = \langle f, g, h \rangle$ a vector field in $\mathbb{R}^3$ and D a region in $\mathbb{R}^3$ enclosed by a surface S then under appropriate conditions

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \nabla \cdot \mathbf{F}\, dV$$

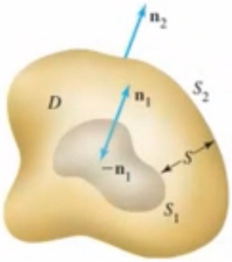
where $\mathbf{n}$ is the outward unit normal vector on S.



- **Divergence Theorem for Hollow Regions:** Given $\mathbf{F} = \langle f, g, h \rangle$ a vector field in $\mathbb{R}^3$ and D a region in $\mathbb{R}^3$ bounded by two oriented surfaces $S_1$and S_2 then under appropriate conditions

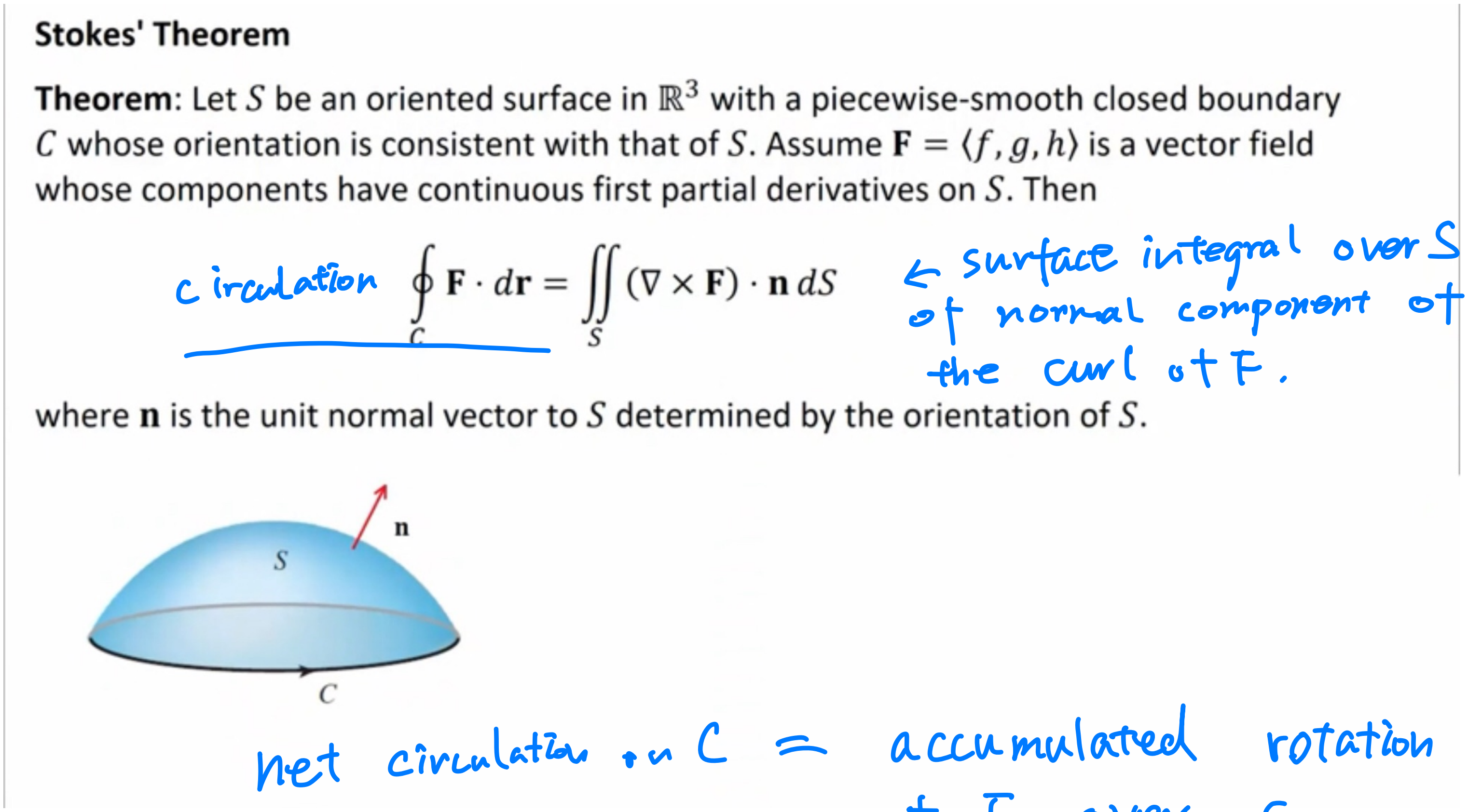
$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS - \iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1\, dS$$

where $\mathbf{n_1}$ and $\mathbf{n_2}$ are the respective outward unit normal vectors.



12.7.

Stokes' Theorem



Stokes' Theorem

Theorem: Let S be an oriented surface in $\mathbb{R}^3$ with a piecewise-smooth closed boundary C whose orientation is consistent with that of S. Assume $\mathbf{F} = \langle f, g, h \rangle$ is a vector field whose components have continuous first partial derivatives on S. Then

$$\underbrace{\text{circulation} \quad \oint_C \mathbf{F} \cdot d\mathbf{r}} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$

← surface integral over S of normal component of the curl of F.

where $\mathbf{n}$ is the unit normal vector to S determined by the orientation of S.

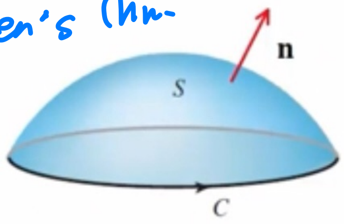
net circulation on C = accumulated rotation of F over S.

Stokes' Theorem

Stokes' Theorem is the three-dimensional version of the circulation form of Green's Theorem.

3D- Green's Thm.

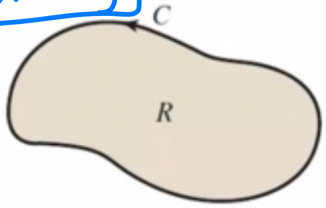
$$\underline{\oint_C \mathbf{F}\cdot d\mathbf{r}} = \iint_S \underbrace{(\nabla\times\mathbf{F})\cdot\mathbf{n}}_{\text{3D Curl}}\, dS$$



Recall that if C is a closed simple piecewise-smooth oriented curve in the xy-plane enclosing a simply connected region R, and $\mathbf{F} = \langle f, g\rangle$ is a differentiable vector field on R then Green's Theorem says

$$\underline{\oint_C \mathbf{F}\cdot d\mathbf{r}} = \boxed{\iint_R \underbrace{\boxed{(g_x - f_y)}}_{\text{2D Curl.}}\, dA.}$$

$$\iint_R \boxed{g_x - f_y}\, dA.$$

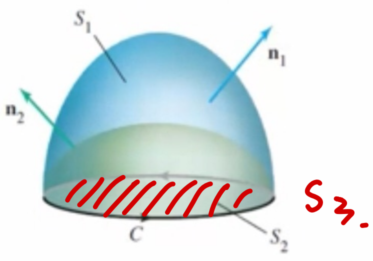


$$\boxed{(-)\iint_R f_y - g_x\, dA}\ X.$$

Important Note about Stokes' Theorem

If a closed curve C is the boundary of two different smooth oriented surfaces S_1 and S_2 which both have orientation consisent with that of C, then the integrals of $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$ on the two sufaces are equal.

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_{S_2} (\underline{\nabla \times \mathbf{F}}) \cdot \mathbf{n}\, dS$$



$$\nabla \times F = (\underline{\qquad})\vec{i} + (\underline{\qquad})\vec{j} + (g_x - f_y)\vec{k}$$

$$\uparrow \boxed{\vec{n} = \langle 0.0, 1\rangle.}$$

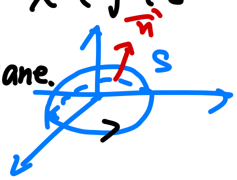
$$(\nabla \times F) \cdot \vec{n} = \boxed{g_x - f_y.}$$

$$\iint_{S_3.} \underline{g_x - f_y}\; d\underline{S}.$$

example) ① Verify the Stoke's Thm.

$F = \langle y, -x, 5 \rangle$ $\quad$ S = upper half of sphere $x^2+y^2+z^2=1$

$\quad\ \ f \quad\ g \quad\ h.$



C = circle $x^2+y^2=1$ on xy-plane.

↳ cc. oriented.

$\vec{n}$ points upward (outward)

$$\text{LHS}\ \boxed{\oint_C F\cdot dr} \overset{?}{=} \iint_S \text{curl}F\cdot n\, dS \quad = -2\pi$$

$$\text{Curl}(F) = \nabla\times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 5 \end{vmatrix} = (0-0)\vec{i} + -(0-0)\vec{j} + \left(\frac{\partial(-x)}{\partial x} - \frac{\partial y}{\partial y}\right)\vec{k}$$

$$= 0\vec{i} + 0\vec{j} + (-1-1)\vec{k}$$

$$= \langle 0.\ 0,\ -2 \rangle.$$

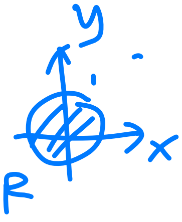
$$z = \sqrt{1-x^2-y^2}$$

$$\vec{n} = \langle -z_x, -z_y, 1 \rangle$$

$$= \left\langle \frac{-x}{\sqrt{1-x^2-y^2}}, -\frac{-y}{\sqrt{1-x^2-y^2}}, 1 \right\rangle = \underline{\underline{\left\langle \frac{x}{z}, \frac{y}{z}, 1 \right\rangle}}.$$

$$\iint_S \text{curF}\cdot\vec{n}\, dS = \iint_R \langle 0,0,-2\rangle\cdot\left\langle \frac{x}{z}, \frac{y}{y}, 1 \right\rangle dA$$

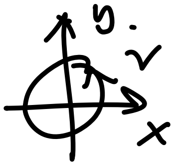
$$= \iint_R -2\, dA. = -2\cdot\pi$$



where R

$$\iint_R dA = \pi(1)^2 = \pi.$$

The line integral $\oint_C F \cdot d\underline{r}$.



$\vec{r}$ = circle of radius 1. on xy-plane $\boxed{z=0.}$

$= \langle \cos t.\ \sin t,\ 0 \rangle,\quad 0 \le t \le 2\pi$

$\vec{F} = \langle y.\ -x,\ 5 \rangle.\ \boxed{\vec{F}(r(t))} = \langle \underset{y}{\sin t},\ \underset{-x}{-\cos t},\ 5 \rangle.$

$r'(t) = \langle -\sin t,\ \cos t,\ 0 \rangle.$

$$\oint_C F\, dr = \oint_0^{2\pi} F \cdot r'\, dt = \oint_0^{2\pi} \underline{\langle \sin t, -\cos t, 5 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle\, dt}$$

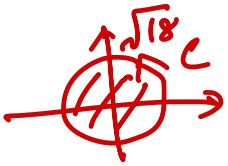
$$= \oint_0^{2\pi} -\sin^2 t - \cos^2 t + 0\, dt.$$

$$= \int_0^{2\pi} -1\ dt$$

$$= \boxed{-2\pi.}$$

stoke's Thm V.

example 2) Evaluate the line integral $\oint_C F \cdot dr$ by using the surface integral in Stoke's Thm with an appropriate choice of S. Assume that C has cc orientation when view from above.



$$F = \langle -9y, -z, x \rangle.$$

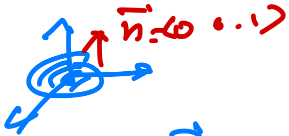
$$C = \text{circle } x^2+y^2=18. \quad \text{on the plane } z=0$$

$$\oint_C F \cdot dr = ?$$

$r(u,v) = \langle u\cos v, u\sin v, 0 \rangle.$

$t_u \times t_v = \langle 0.0, u \rangle.$

① surface S: $x^2+y^2=18$ on $z=0$. (Explicit)



$\vec{n} = \langle 0.0.1 \rangle.$

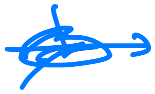
$$\text{curl } F = \nabla \times F = \begin{vmatrix} i & j & \vec{k} \\ \cancel{\frac{\partial}{\partial x}} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -9y & -z & x \end{vmatrix} = (0-(-1))\,i + (-1-0)\,\vec{j} + \left(0 - \frac{\partial(-9y)}{\partial y}\right)\vec{k}$$

$$= 1\vec{i} - 1\vec{j} + 9\vec{k}$$

$$\iint_S \text{curl} F \cdot n\,dS = \iint_R \langle 1,-1,9 \rangle \cdot \langle 0.0.1 \rangle\, dA = \langle 1,-1,9 \rangle$$

$$= \iint_R 9\, dA = 9\iint_R dA = 9\pi(\sqrt{18})^2 = 162\pi$$

② Parametric



$r(u,v) = \langle u\cos v,\ u\sin v,\ 0 \rangle$

$0 \le u \le \sqrt{18}$

$0 \le v \le 2\pi$

$t_u \times t_v = \langle 0, 0, u \rangle$.

$\text{curl}(F) = \langle 1, -1, 9 \rangle$.

$$\iint_S \text{curl}F \cdot \vec{n}\, dS = \int_0^{2\pi}\int_0^{\sqrt{18}} \langle 1, -1, 9\rangle \langle 0, 0, u\rangle\, du\, dv.$$

$$= \int_0^{2\pi}\int_0^{\sqrt{18}} 9u\, du\, dv$$

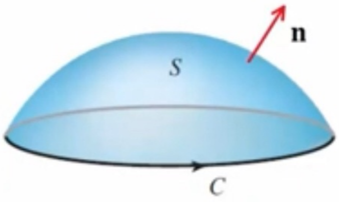
$$= 2\pi \ \frac{9}{2}u^2 \Big|_0^{\sqrt{18}}$$

$$= 9\pi(\sqrt{18})^2 = 162\pi.$$

Green's Thm

↓

- **Stokes' Theorem:** Given $\mathbf{F} = \langle f, g, h\rangle$ a vector field in $\mathbb{R}^3$ and S an oriented surface in $\mathbb{R}^3$ with closed boundary C whose orientation is consistent with that of S then under appropriate conditions



$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$$

↓

$\oint \omega = \int d\omega.$

where $\mathbf{n}$ is the unit normal vector to S determined by the orientation of S.

- Stokes' Theorem can be used in either direction. That is, it can be used to help evaluate a line integral or to help evaluate a surface integral.

- If C bounds two surfaces then choose the easier one when evaluating the surface integral from Stokes' Theorem!

Divergence Theorem

Divergence Theorem

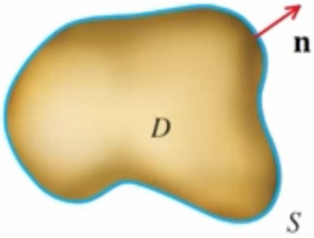
Theorem: Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives in a connected and simply connected region D in $\mathbb{R}^3$ enclosed by an oriented surface S. Then

flux of F across the boundary surface of D.

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \nabla\cdot\mathbf{F}\,dV$$

triple integral over the divergence of F over D.

where $\mathbf{n}$ is the outward unit normal vector to S.



net flow of F across the boundary S.

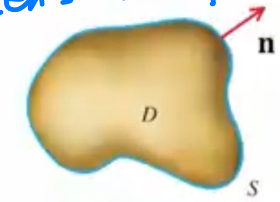
the cumulative expansion or contraction of F over D.

Divergence Theorem

The Divergence Theorem is the three-dimensional version of the flux form of Green's Theorem.

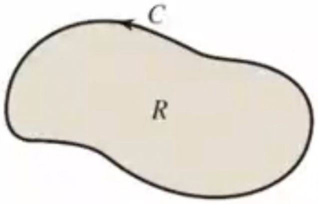
3D - "Green's Th" flux form

$$\underbrace{\boxed{\iint_S \mathbf{F}\cdot\mathbf{n}\,dS}}_{\text{flux}} = \iiint_D \underbrace{\nabla\cdot\mathbf{F}}_{\text{3D-div}}\,dV$$



Recall that if C is a closed simple piecewise-smooth oriented curve in the xy-plane enclosing a simply connected region R, and $\mathbf{F} = \langle f, g\rangle$ is a differentiable vector field on R then Green's Theorem says

$$\underbrace{\boxed{\oint_C \mathbf{F}\cdot\mathbf{n}\,ds}}_{\text{flux}} = \iint_R \underbrace{(f_x + g_y)}_{\text{2D-div}}\,dA$$



ex.) Verify Divergence Thm

$F = \langle 3x, 2y, 3z \rangle \qquad D = \{(x, y, z)\ x^2+y^2+z^2 \le 4\}$

$$\iiint_D \operatorname{div} F \, dV = \iiint_D 3+2+3 \, dV = \iiint_D 8 \, dV = 8 \iiint_D dV = 8(\text{volume of } D)$$

$$= 8\left(\frac{4}{3}\pi 2^3\right) = \frac{2^8}{3}\pi$$

$\iint F \cdot n \, dS$

② Parametric $\quad F = \langle 3x, 2y, 3z \rangle$

$r(u,v) = \langle 2\sin u \cos v, 2\sin u \sin v, 2\cos u \rangle \quad 0 \le u < \pi$

$\vec{n} = \langle 4\sin^2 u \cos v, 4\sin^2 u \sin v, 4\sin u \cos u \rangle \quad 0 \le v \le 2\pi$

$$\iint_S F \cdot n \, dS = \iint_R \langle 6\sin u \cos v, 4\sin u \sin v, 6\cos u \rangle \cdot \langle 4\sin^2 u \cos v, 4\sin^2 u \sin v, 4\sin u \cos u \rangle \, dA$$

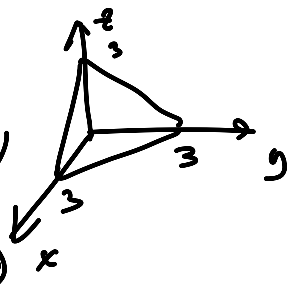
$$= \iint_R 24\sin^3 u \cos^2 v + 16\sin^3 u \sin^2 v + 24\sin u \cos^2 u \, dA$$

... we can integrate this using the trig indentity ... and get $\frac{2^8\pi}{3}$

ex) Compute the net outward flux on $F = \langle -x, 3y. 2z \rangle$ across S. where S is the boundary of $x+y+z=3$ on the first octend

$$\iint_S F\cdot n\, ds = \iiint_D \nabla\cdot F\, dV = \iiint_D -1+3+2\, dV = 4\iiint_D dV$$

$$= 4(\text{volume of D})$$



what we have is a trirectangular tetrachedron and the volume is $\frac{abc}{6}$ and in our case $\frac{3\cdot3\cdot3}{6} = \frac{9}{2}$

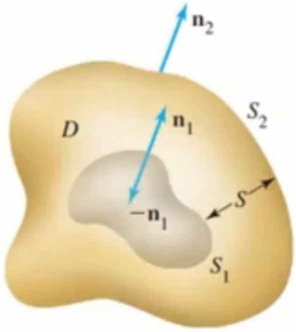
so $\iiint_D 4\, dV = \frac{9}{2}\cdot 4 = 18.$

If we want to compute the flux on the surface boundary the tretrahedron — compute the flux on 4 surface where you might not want to do.

Divergence Theorem for Hollow Regions

Theorem: Suppose the vector field $\mathbf{F}$ satisfies the conditions of the Divergence Theorem on a region D in $\mathbb{R}^3$ bounded by two oriented surfaces S_1 and S_2 where S_1 lies within S_2. Let S be the entire boundary of D $(S = S_1 \cup S_2)$ and let $\mathbf{n_1}$ and $\mathbf{n_2}$ be the outward unit normal vectors for S_1 and S_2 respectively. Then

$$\iiint_D \underbrace{\nabla \cdot \mathbf{F}}_{\text{divergence}}\, dV = \underbrace{\iint_S \mathbf{F} \cdot \mathbf{n}\, dS}_{\text{flux}} = \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS - \iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1\, dS.$$



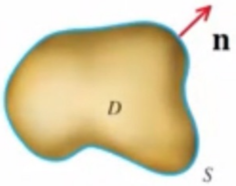
This form of the Divergence Theorem is applicable to vector fields that are not differentiable at the origin, as is the case with some important radial vector fields of the form:

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p}.$$

Summary

- **Divergence Theorem:** Given $\mathbf{F} = \langle f, g, h \rangle$ a vector field in $\mathbb{R}^3$ and D a region in $\mathbb{R}^3$ enclosed by a surface S then under appropriate conditions

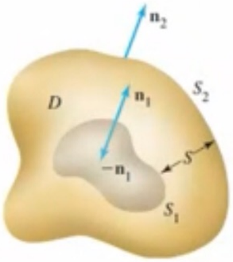
$$\iint\limits_{S} \mathbf{F} \cdot \mathbf{n} \, dS = \iiint\limits_{D} \nabla \cdot \mathbf{F} \, dV$$



where $\mathbf{n}$ is the outward unit normal vector on S.

- **Divergence Theorem for Hollow Regions:** Given $\mathbf{F} = \langle f, g, h \rangle$ a vector field in $\mathbb{R}^3$ and D a region in $\mathbb{R}^3$ bounded by two oriented surfaces $S_1$and S_2 then under appropriate conditions

$$\iiint\limits_{D} \nabla \cdot \mathbf{F} \, dV = \iint\limits_{S} \mathbf{F} \cdot \mathbf{n} \, dS = \iint\limits_{S_2} \mathbf{F} \cdot \mathbf{n}_2 \, dS - \iint\limits_{S_1} \mathbf{F} \cdot \mathbf{n}_1 \, dS$$



where $\mathbf{n_1}$ and $\mathbf{n_2}$ are the respective outward unit normal vectors.